On a Fuzzy Logistic Difference Equation

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Abstract: This paper is concerned with the existence, uniqueness and asymptotic behavior of the positive solutions of a fuzzy Logistic difference equation

$$x_{n+1} = A + Bx_{n-1}e^{-x_n},\ n = 0, 1, \cdots,$$

where (x_n) is a sequence of positive fuzzy number, A, B are positive fuzzy numbers and the initial conditions x_{-1}, x_0 are positive fuzzy numbers. Moreover an illustrative example is given to demonstrate the effectiveness of the results obtained.

Key–Words: Fuzzy Logistic difference equation, Equilibrium point, Bounded, Persistence

1 Introduction

It is well known that difference equation appears naturally as discrete analogous and as numerical solutions of differential equations and delay differential equation having many applications in economics, biology, computer science, control engineering. The study of asymptotic stability and oscillatory properties of solutions of difference equations is extremely useful in the behavior of mathematical models of various biological systems and other applications. This is due to the fact that difference equations are appropriate models for describing situations where the variable is assumed to take only a discrete set of values and they arise frequently in the study of biological models, in the formulation and analysis of discrete time systems, etc.

Recently there has been a lot of work concerning the asymptotic behavior, the periodicity, and the boundedness of nonlinear difference equation and system of nonlinear difference equations(see, for example [1, 5, 6, 7, 8, 9, 10, 13, 14] and the references therein).

EI-Metwally et al.[7] investigated the asymptotic behavior of population model:

$$x_{n+1} = \alpha + \beta x_{n-1}e^{-x_n},\ n = 0, 1, \cdots, \quad (1)$$

where α is the immigration rate and β is the population growth rate.

Fuzzy set theory is a powerful tool for modeling uncertainty and for processing vague or subjective information in mathematical model. Particularly, the use of fuzzy difference equations is a natural way to model the dynamical systems with embedded uncertainty. Fuzzy difference equation is a difference equation where constants and the initial values are fuzzy numbers, and its' solutions are sequences of fuzzy numbers. Recently there is an increasing interest in the study of fuzzy difference equation (see, for example [2, 3, 4, 11, 12, 15, 16, 17, 18, **?**, 21]). In this paper we study the fuzzy analogs of (1)

$$x_{n+1} = A + Bx_{n-1}e^{-x_n},\ n = 0, 1, \cdots, \quad (2)$$

where (x_n) is a sequence of positive fuzzy numbers, A, B and the initial values x_{-1}, x_0 are positive fuzzy numbers.

For readers convenience, we need some definitions:

A is said to be a fuzzy number if $A : R \to [0,1]$ satisfies the below (i)-(iv)

(i) A is normal, i.e. there exists an $x \in R$ such that $A(x) = 1$;

(ii) A is fuzzy convex, i.e. for all $t \in [0,1]$ and $x_1, x_2 \in R$ such that

$$A(tx_1 + (1-t)x_2) \geq \min\{A(x_1), A(x_2)\};$$

(iii) A is upper semi-continuous;

(iv) The support of A,

$$\text{supp}A = \overline{\bigcup_{\alpha\in(0,1]} [A]_\alpha} = \overline{\{x : A(x) > 0\}}$$

is compact.

The α-cuts of A are denoted by

$$[A]_\alpha = \{x \in R : A(x) \geq \alpha\}, \ \ \alpha \in [0,1],$$

it is clear that the $[A]_\alpha$ are closed interval. We say that a fuzzy number is positive if $\text{supp}A \subset (0,\infty)$.

It is obvious that if A is a positive real number then A is a fuzzy numbers and $[A]_\alpha = [A,A], \alpha \in (0,1]$. Then we say that A is a trivial fuzzy number.

Let A, B be fuzzy numbers with $[A]_\alpha = [A_{l,\alpha}, A_{r,\alpha}], [B]_\alpha = [B_{l,\alpha}, B_{r,\alpha}], \alpha \in (0,1]$. We define a norm on fuzzy numbers space as follows:

$$\|A\| = \sup_{\alpha\in(0,1]} \max\{|A_{l,\alpha}|, |A_{r,\alpha}|\}$$

We take the following metric :

$$D(A,B) = \sup_{\alpha\in(0,1]} \max\{|A_{l,\alpha} - B_{l,\alpha}|, |A_{r,\alpha} - B_{r,\alpha}|\}$$

The fuzzy analog of the boundedness and persistence (see [4, 11]) as follows: we say that a sequence of positive fuzzy numbers x_n persists (resp. is bounded) if there exists a positive real number M (resp. N) such that

$$\text{supp}x_n \subset [M,\infty)(\text{resp. supp}x_n \subset (0,N]), n \in \mathbb{N},$$

x_n is bounded and persists if there exist positive real numbers $M, N > 0$ such that

$$\text{supp}x_n \subset [M,N], \ \ n = 1,2,\cdots.$$

$x_n, n = 1,2,\cdots$, is an unbounded sequence if the norm $\|x_n\|, n = 1,2,\cdots$, is an unbounded sequence.

x_n is a positive solution of (2) if x_n is a sequence of positive fuzzy numbers which satisfies (2). We say a positive fuzzy number x is a positive equilibrium for (2) if

$$x = A + Bxe^{-x}.$$

Let (x_n) be a sequence of positive fuzzy numbers and x is a positive fuzzy number, Suppose that

$$[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \ n = 0,1,2,\cdots, \ \ \alpha \in (0,1], \tag{3}$$

and

$$[x]_\alpha = [L_\alpha, R_\alpha], \ \ \alpha \in (0,1]. \tag{4}$$

The sequence (x_n) converges to x with respect to D as $n \to \infty$ if $\lim_{n\to\infty} D(x_n, x) = 0$.

Suppose that (2) has a unique positive equilibrium x. We say that the positive equilibrium x of (2) is stable if for every $\varepsilon > 0$ there exists a $\delta = \delta(\varepsilon) > 0$ such that for every positive solution x_n of (2), which satisfies $D(x_{-i}, x) \leq \delta, i = 0,1$, we have $D(x_n, x) \leq \varepsilon$ for all $n > 0$.

Moreover, we say that the positive equilibrium x of (2) is asymptotically stable, if it is stable and every positive solution of (2) tends to the positive equilibrium of (2) with respect to D as $n \to \infty$.

The purpose of this paper is to study existence of the positive solutions of (2). Furthermore, we give some conditions so that every positive solution of (2) is boundedness and persistence. Finally, under some conditions we prove that (2) has a unique positive equilibrium x which is asymptotic stable.

2 Main results

2.1 Existence of the positive solution

Firstly we study the existence of the positive solutions of (2). We need the following lemma.

Lemma 1 *[12] Let $f :: R^+ \times R^+ \times R^+ \times R^+ \to R^+$ be continuous, A, B, C, D are fuzzy numbers, Then, for $\forall\alpha \in (0,1]$,*

$$[f(A,B,C,D)]_\alpha = f([A]_\alpha, [B]_\alpha, [C]_\alpha, [D]_\alpha). \tag{5}$$

Lemma 2 *[19] Let $u \in E^\sim$, write $[u]_\alpha = [u_-(\alpha), u_+(\alpha)], \alpha \in (0,1]$. Then $u_-(\alpha)$ and $u_+(\alpha)$ can be regarded as functions on $(0,1]$, which satisfy*

(i) $u_-(\alpha)$ is nondecreasing and left continuous;

(ii) $u_+(\alpha)$ is nonincreasing and left continuous;

(iii) $u_-(1) \leq u_+(1)$.

Conversely for any functions $a(\alpha)$ and $b(\alpha)$ defined on $(0,1]$ which satisfy $(i) - (iii)$ in the above, there exists a unique $u \in E^\sim$ such that $[u]_\alpha = [a(\alpha), b(\alpha)]$ for any $\alpha \in (0,1]$.

Theorem 3 *Consider equation (2) where A, B are positive fuzzy numbers. Then for any positive fuzzy numbers x_{-1}, x_0, there exists a unique positive solution x_n of (2) with initial conditions x_{-1}, x_0.*

Proof: The proof is similar to Proposition 2.1 in [12]. Suppose that there exists a sequence of fuzzy numbers x_n satisfying (2) with initial conditions x_{-1}, x_0. Consider the $\alpha-$cuts, $\alpha \in (0,1]$, $n = 0, 1, 2, \cdots$,

$$\begin{cases} [x_n]_\alpha &= [L_{n,\alpha}, R_{n,\alpha}], \\ [A]_\alpha &= [A_{l,\alpha}, A_{r,\alpha}], \\ [B]_\alpha &= [B_{l,\alpha}, B_{r,\alpha}]. \end{cases} \tag{6}$$

It follows from (2), (6) and Lemma 1 that

$$\begin{aligned} [x_{n+1}]_\alpha &= [L_{n+1,\alpha}, R_{n+1,\alpha}] \\ &= [A + Bx_{n-1}e^{-x_n}]_\alpha \\ &= [A]_\alpha + [B]_\alpha [x_{n-1}]_\alpha \times [e^{-x_n}]_\alpha \\ &= \big[A_{l,\alpha} + B_{l,\alpha} L_{n-1,\alpha} e^{-R_{n,\alpha}}, \\ &\quad A_{r,\alpha} + B_{r,\alpha} R_{n-1,\alpha} e^{-L_{n,\alpha}}\big] \end{aligned}$$

from which we have that for $n = 0, 1, 2, \cdots, \alpha \in (0, 1]$,

$$\begin{cases} L_{n+1,\alpha} &= A_{l,\alpha} + B_{l,\alpha} L_{n-1,\alpha} e^{-R_{n,\alpha}}, \\ R_{n+1,\alpha} &= A_{r,\alpha} + B_{r,\alpha} R_{n-1,\alpha} e^{-L_{n,\alpha}} \end{cases} \tag{7}$$

Then it is obvious that for any initial condition $(L_{-i,\alpha}, R_{-i,\alpha}), i = 0, 1, \alpha \in (0, 1]$, there exists a unique solution $(L_{n,\alpha}, R_{n,\alpha})$. Now we prove that $[L_{n,\alpha}, R_{n,\alpha}], \alpha \in (0, 1]$, where $(L_{n,\alpha}, R_{n,\alpha})$ is the solution of system (7) with initial conditions $(L_{-i,\alpha}, R_{-i,\alpha}), i = 0, 1$, determines the solution x_n of (2) with the initial conditions $x_{-i}, i = 0, 1$, such that, for $n = 0, 1, 2, \cdots$,

$$[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \ \ \alpha \in (0, 1]. \tag{8}$$

From reference [20] and since A, B, x_{-1}, x_0 are positive fuzzy numbers for any $\alpha_1, \alpha_2 \in (0, 1], \alpha_1 \le \alpha_2$ we have

$$\begin{cases} 0 < A_{l,\alpha_1} \le A_{l,\alpha_2} \le A_{r,\alpha_2} \le A_{r,\alpha_1} \\ 0 < B_{l,\alpha_1} \le B_{l,\alpha_2} \le B_{r,\alpha_2} \le B_{r,\alpha_1} \\ 0 < L_{-1,\alpha_1} \le L_{-1,\alpha_2} \le R_{-1,\alpha_2} \le R_{-1,\alpha_1} \\ 0 < L_{0,\alpha_1} \le L_{0,\alpha_2} \le R_{0,\alpha_2} \le R_{0,\alpha_1} \end{cases} \tag{9}$$

We claim that, for $n = -1, 0, 1, 2, \cdots$.

$$L_{n,\alpha_1} \le L_{n,\alpha_2} \le R_{n,\alpha_2} \le R_{n,\alpha_1}. \tag{10}$$

We prove it by induction. It is obvious from (9) that (10) holds true for $n = -1, 0$. Suppose that (10) are true for $n \le k, k \in \{1, 2, \cdots\}$. Then, from (7), (9) and (10), for $n \le k$, it follows that

$$\begin{aligned} L_{k+1,\alpha_1} &= A_{l,\alpha_1} + B_{l,\alpha_1} L_{k-1,\alpha_1} e^{-R_{k,\alpha_1}} \\ &\le A_{l,\alpha_2} + B_{l,\alpha_2} L_{k,\alpha_2} e^{-R_{k,\alpha_2}} = L_{k+1,\alpha_2} \\ &= A_{l,\alpha_2} + B_{l,\alpha_2} L_{k,\alpha_2} e^{-R_{k,\alpha_2}} \\ &\le A_{r,\alpha_2} + B_{r,\alpha_2} L_{k,\alpha_2} e^{-R_{k,\alpha_2}} = R_{k+1,\alpha_2} \\ &= A_{r,\alpha_2} + B_{r,\alpha_2} L_{k,\alpha_2} e^{-R_{k,\alpha_2}} \\ &\le A_{r,\alpha_1} + B_{r,\alpha_1} L_{k,\alpha_1} e^{-R_{k,\alpha_1}} = R_{k+1,\alpha_1} \end{aligned}$$

Therefore (10) are satisfied. Moreover from (7) we have, for $\forall \alpha \in (0, 1]$,

$$\begin{cases} L_{1,\alpha} = A_{l,\alpha} + B_{l,\alpha} L_{-1,\alpha} e^{-R_{0,\alpha}}, \\ R_{1,\alpha} = A_{r,\alpha} + B_{r,\alpha} R_{-1,\alpha} e^{-L_{0,\alpha}}. \end{cases} \tag{11}$$

Since A, B, x_{-1}, x_0 are positive fuzzy numbers, then we have that $A_{l,\alpha}$, $A_{r,\alpha}$, $B_{l,\alpha}$, $B_{r,\alpha}$, $L_{-l,\alpha}$, $R_{-1,\alpha}$, $L_{0,\alpha}$, $R_{0,\alpha}$ are left continuous. So from (11) we have that $L_{1,\alpha}, R_{1,\alpha}$ are also left continuous. By induction we can get that $L_{n,\alpha}, R_{n,\alpha}, n = 1, 2, \cdots$, are left continuous.

Now we prove that the support of x_n, $\text{supp}x_n = \overline{\bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}]}$ is compact. It is sufficient to prove that $\bigcup_{\alpha \in (0,1]} [L_{n,\alpha}, R_{n,\alpha}]$ is bounded. Let $n = 1$, since A, B, x_{-1}, x_0 are positive fuzzy numbers, there exist constants $M_A > 0, N_A > 0, M_B > 0, N_B > 0, M_{-1} > 0, N_{-1} > 0, M_0 > 0, N_0 > 0$ such that for all $\alpha \in (0, 1]$,

$$\begin{cases} [A_{l,\alpha}, A_{r,\alpha}] & \subset [M_A, N_A], \\ [B_{l,\alpha}, B_{r,\alpha}] & \subset [M_B, N_B], \\ [L_{-l,\alpha}, R_{-1,\alpha}] & \subset [M_{-1}, N_{-1}], \\ [L_{0,\alpha}, R_{0,\alpha}] & \subset [M_0, N_0] \end{cases} \tag{12}$$

Hence from (11) and (12) we can easily get, for $\forall \alpha \in (0, 1]$,

$$\begin{aligned} [L_{1,\alpha}, R_{1,\alpha}] \subset \big[& M_A + M_B M_{-1} e^{-N_0}, \\ & N_A + N_B N_{-1} e^{-M_0}\big]. \end{aligned} \tag{13}$$

From which it is obvious that

$$\bigcup_{\alpha \in (0,1]} [L_{1,\alpha}, R_{1,\alpha}] \subset \big[M_A + M_B M_{-1} e^{-N_0}, N_A + N_B N_{-1} e^{-M_0}\big] \tag{14}$$

Therefore (14) implies that $\overline{\bigcup_{\alpha\in(0,1]}[L_{1,\alpha},R_{1,\alpha}]}$ is compact and $\overline{\bigcup_{\alpha\in(0,1]}[L_{1,\alpha},R_{1,\alpha}]} \subset (0,\infty)$. Deducing inductively we can easily follow that $\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha},R_{n,\alpha}]}$ is compact, and

$$\overline{\bigcup_{\alpha\in(0,1]}[L_{n,\alpha},R_{n,\alpha}]} \subset (0,\infty),\ n=1,2,\cdots. \quad (15)$$

Therefore, (10), (15) and since $L_{n,\alpha}, R_{n,\alpha}$ are left continuous we have that $[L_{n,\alpha},R_{n,\alpha}]$ determines a sequence of positive fuzzy numbers x_n such that (8) holds.

We prove now that x_n is the solution of (2) with initial condition x_{-1}, x_0. Since for all $\alpha\in(0,1]$,

$$\begin{aligned}[x_{n+1}]_\alpha &= [L_{n+1,\alpha},R_{n+1,\alpha}] \\ &= \big[A_{l,\alpha}+B_{l,\alpha}L_{n-1,\alpha}e^{-R_{n,\alpha}}, \\ &\quad A_{r,\alpha}+B_{r,\alpha}R_{n-1,\alpha}e^{-L_{n,\alpha}}\big] \\ &= \big[A+Bx_{n-1}e^{-x-n}\big]_\alpha\end{aligned}$$

we have that x_n is the solution of (2) with initial condition x_{-1}, x_0.

Suppose that there exists another solution $\overline{x}_n$ of (2) with initial conditions x_{-1}, x_0. Then from arguing as above we can easily prove that,for $n=0,1,2,\cdots$,

$$[\overline{x}_n]_\alpha = [L_{n,\alpha},R_{n,\alpha}], \quad \alpha\in(0,1]. \quad (16)$$

Then from (8) and (16) we have $[x_n]_\alpha=[\overline{x}_n]_\alpha, \alpha\in(0,1], n=0,1,2,\cdots$, from which it follows that $x_n=\overline{x}_n, n=0,1,\cdots$. Thus the proof of the theorem 3 is completed. □

2.2 Boundedness and Permanence

In the following we will study the boundedness and permanence of the fuzzy positive solution of (2). We need the following lemma.

Lemma 4 *(comparison results) Assume that* $\alpha\in(0,\infty), \beta\in[0,\infty)$. *Let* $\{X_n\}_{n=-1}^{\infty}, \{Y_n\}_{n=-1}^{\infty}$ *be sequences of real numbers such that* $X_{-1}\le Y_{-1}$, $X_0\le Y_0$, *and for* $n=0,1,\cdots$,

$$\begin{cases} X_{n+1}\le \alpha X_{n-1}+\beta \\ Y_{n+1}=\alpha Y_{n-1}+\beta \end{cases}$$

Then $X_n\le Y_n$, *for all* $n\ge -1$.

Lemma 5 *Consider the system of the difference equations, for* $n=0,1,\cdots$,

$$\begin{cases} y_{n+1}=p+cy_{n-1}e^{-z_n} \\ z_{n+1}=q+dz_{n-1}e^{-y_n} \end{cases} \quad (17)$$

where p,q,c,d *are positive real numbers and the initial values* $y_{-i}, z_{-i}(i=0,1)$, *are positive real numbers. If*

$$c<e^q, \quad d<e^p \quad (18)$$

Then system (17) is bounded and persists.

Proof: Let $\{(y_n,z_n)\}_{n=-1}^{\infty}$ be a positive solution of system (17). Then it follows from system (17) that, for $n\ge 0$,

$$\begin{cases} y_{n+1}=p+cy_{n-1}e^{-z_n}>p \\ z_{n+1}=q+dz_{n-1}e^{-y_n}>q \end{cases} \quad (19)$$

Thus

$$\begin{cases} y_{n+1}=p+cy_{n-1}e^{-z_n}<p+ce^{-q}y_{n-1}, \\ z_{n+1}=q+dz_{n-1}e^{-y_n}<q+de^{-p}z_{n-1}. \end{cases} \quad (20)$$

Now we consider the initial value problem, for $n=1,2,\cdots$,

$$Y_{n+1}=p+ce^{-q}Y_{n-1}, \ \ Z_{n+1}=q+de^{-p}Z_{n-1},$$

with initial conditions $y_i\le Y_i, z_i\le Z_i (i=0,1)$, and so it follows from Lemma 4 that

$$y_n\le Y_n, \ \ z_n\le Z_n, \ n\ge -1.$$

Observe that

$$\lim_{n\to\infty} Y_n=\frac{p}{1-ce^{-q}}, \ \lim_{n\to\infty} Z_n=\frac{q}{1-de^{-p}}.$$

and then

$$\lim_{n\to\infty}\sup\, y_n\le\frac{p}{1-ce^{-q}}, \ \lim_{n\to\infty}\sup\, z_n\le\frac{q}{1-de^{-p}}.$$

Therefore $\{(y_n,z_n)\}_{n=-1}^{\infty}$ is bounded and persists. and the proof is completed. □

Theorem 6 *Consider fuzzy difference equation (2), where* A, B *and initial values* x_{-1}, x_0 *are positive fuzzy numbers. If for all* $\alpha\in(0,1]$,

$$B_{r,\alpha}<e^{A_{l,\alpha}}. \quad (21)$$

Then every positive solution of (2) is bounded and persists.

Proof: Let x_n be positive solution of Eq.(2) such that (8) holds. From (7) it is obvious that, for $n=1,2,\cdots,\alpha\in(0,1]$,

$$A_{l,\alpha}\le L_{n,\alpha}, \quad A_{r,\alpha}\le R_{n,\alpha}. \quad (22)$$

From Lemma 4 and (19), it follows that, for $n = 1, 2, \cdots$,

$$\begin{cases} L_{n,\alpha} \leq \frac{A_{l,\alpha}}{1-B_{l,\alpha}e^{-A_{r,\alpha}}}, \\ \\ R_{n,\alpha} \leq \frac{A_{r,\alpha}}{1-B_{r,\alpha}e^{-A_{l,\alpha}}}, \end{cases} \tag{23}$$

Hence from (12), (20) and (21) it is obvious that, for $n \geq 1$. $\alpha \in (0, 1]$,

$$[L_{n,\alpha}, R_{n,\alpha}] \subset \left[M_A, \frac{N_A}{1 - N_B e^{-M_A}}\right]. \tag{24}$$

From which we get, for $n \geq 1$,

$$\bigcup_{\alpha\in(0,1]} [L_{n,\alpha}, R_{n,\alpha}] \subset \left[M_A, \frac{N_A}{1 - N_B e^{-M_A}}\right],$$

and so

$$\overline{\bigcup_{\alpha\in(0,1]} [L_{n,\alpha}, R_{n,\alpha}]} \subset \left[M_A, \frac{N_A}{1 - N_B e^{-M_A}}\right].$$

i.e., the positive solution of (2) is bounded and persists.

2.3 Dynamic of solution of Eq.(2)

Lemma 7 *Consider the system of difference equation (17), where $p, q, c, d,$ are positive real numbers. If (18) holds. Then there exists a unique positive equilibrium $(\bar{y}, \bar{z})$ such that*

$$p < \bar{y} < \frac{p}{1 - ce^{-q}}, \quad q < \bar{z} < \frac{q}{1 - de^{-p}}. \tag{25}$$

Proof: Let $(\bar{y}, \bar{z})$ be the solution of the following systems.

$$y = p + cye^{-z}, \quad z = q + dze^{-y} \tag{26}$$

Set

$$f(z) = q + dze^{-\frac{p}{1-ce^{-z}}} - z \tag{27}$$

Then

$$f(q) = dqe^{\frac{p}{ce^{-q}-1}} > 0, \quad \lim_{z\to+\infty} f(z) = -\infty. \tag{28}$$

and

$$f'(z) = de^{-\frac{p}{1-ce^{-z}}} - dz\frac{pce^{-z}}{(1 - ce^{-z})^2}e^{-\frac{p}{1-ce^{-z}}} - 1 < 0. \tag{29}$$

It follows from (26) and (16) that (25) has exactly one solution $\bar{z} > q$.

On the other hand, set

$$g(y) = p + cye^{-\frac{q}{1-de^{-y}}} - y \tag{30}$$

Similarly it can easily prove that (28) has exactly one solution $\bar{y} > p$. Noting (18) and (24) it follows that $\bar{y} < \frac{p}{1-ce^{-q}}, \bar{z} < \frac{q}{1-de^{-p}}$. The proof is completed. □

Theorem 8 *Consider the system (17), where p, q, c, d are positive real numbers and the initial values $y_{-i}, z_{-i}(i = 0, 1)$, are positive real numbers. Assume that (18) and the following condition hold*

$$\max \left\{ \left((1 + \frac{p}{1 - ce^{-q}})ce^{-q} \right)^{\frac{1}{2}}, \right.$$
$$\left. (1 + \frac{q}{1 - de^{-p}})de^{-p} \right\} < 1 \tag{31}$$

Then the positive equilibrium $(\bar{y}, \bar{z})$ of system (17) is globally asymptotically stable.

Proof: First we prove that the positive equilibrium $(\bar{y}, \bar{z})$ of (17) is locally asymptotically stable. We can easily obtain that the linearized system of (17) about the positive equilibrium $(\bar{y}, \bar{z})$ is

$$\Psi_{n+1} = D\Psi_n,$$

where

$$D = \begin{pmatrix} 0 & ce^{-\bar{z}} & -c\bar{y}e^{-\bar{z}} & 0 \\ 1 & 0 & 0 & 0 \\ -d\bar{z}e^{-\bar{y}} & 0 & 0 & de^{-\bar{y}} \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Since (18) and (29) hold, we can take a positive number ε such that

$$\begin{aligned} \varepsilon &= \max \left\{ \left((1 + \bar{y})ce^{-\bar{z}} \right)^{\frac{1}{2}}, (1 + \bar{z})de^{-\bar{y}} \right\} \\ &< 1. \end{aligned} \tag{32}$$

Let

$$S = \text{diag}(1, \varepsilon^{-1}, \varepsilon^{-2}, \varepsilon^{-3})$$

We consider the matrix

$$C = (c_{ij}) = S^{-1}DS =$$

$$\begin{pmatrix} 0 & ce^{-\bar{z}}\varepsilon^{-1} & -c\bar{y}e^{-\bar{z}}\varepsilon^{-2} & 0 \\ \varepsilon & 0 & 0 & 0 \\ -d\bar{z}e^{-\bar{y}}\varepsilon^2 & 0 & 0 & de^{-\bar{y}}\varepsilon^{-1} \\ 0 & 0 & \varepsilon & 0 \end{pmatrix}$$

with the norm

$$\|C\| = \max_{1\leq i\leq 4} \left\{ \sum_{j=1}^{4} |c_{ij}| \right\}.$$

Using (30) we can prove that $\|C\| < 1$. Therefore, since $|\lambda_i| < \|C\|$(where $\lambda_i (i = 1, 2, 3, 4)$ are the eigenvalues of D), we have that all eigenvalues of D lie inside the unit disk. This implies that $(\bar{y}, \bar{z})$ is locally asymptotically stable.

Let now (y_n, z_n) be a positive solution of (17). We prove that

$$\lim y_n = \bar{y}, \quad \lim z_n = \bar{z}. \tag{33}$$

Using Lemma 4 we have

$$\begin{cases} \Lambda_1 = \lim_{n\to\infty} \sup y_n < \infty, \\ \Lambda_2 = \lim_{n\to\infty} \sup z_n < \infty. \\ \lambda_1 = \lim_{n\to\infty} \inf y_n > 0, \\ \lambda_2 = \lim_{n\to\infty} \inf z_n > 0. \end{cases} \tag{34}$$

Then from (17) and (32) we take

$$\begin{cases} \Lambda_1 \le \frac{p}{1-ce^{-\lambda_2}}, \lambda_1 \ge \frac{p}{1-ce^{-\Lambda_2}}, \\ \Lambda_2 \le \frac{q}{1-de^{-\lambda_1}}, \lambda_2 \ge \frac{q}{1-de^{-\Lambda_1}}. \end{cases} \tag{35}$$

From (33) it follows that

$$\begin{cases} \Lambda_1\lambda_2 \le \frac{p\lambda_2}{1-ce^{-\lambda_2}},\ \Lambda_2\lambda_1 \ge \frac{p\Lambda_2}{1-ce^{-\Lambda_2}}. \\ \Lambda_2\lambda_1 \le \frac{q\lambda_1}{1-de^{-\lambda_1}}, \lambda_2\Lambda_1 \ge \frac{q\Lambda_1}{1-de^{-\Lambda_1}} \end{cases}$$

from which we take

$$\begin{cases} \frac{q\Lambda_1}{1-de^{-\Lambda_1}} \le \frac{p\lambda_2}{1-ce^{-\lambda_2}}, \\ \frac{p\Lambda_2}{1-ce^{-\Lambda_2}} \le \frac{q\lambda_1}{1-de^{-\lambda_1}} \end{cases} \tag{36}$$

Now we consider the functions

$$f(y) = \frac{qy}{1-de^{-y}},\ \ g(z) = \frac{pz}{1-ce^{-z}}. \tag{37}$$

for

$$y \in \left(p, \frac{p}{1-ce^{-q}}\right), z \in \left(q, \frac{q}{1-de^{-p}}\right).$$

Then from (35), it follows that

$$\begin{cases} f'(y) = \frac{q(1-de^{-y}(1+y))}{(1-de^{-y})^2}, \\ g'(z) = \frac{p(1-ce^{-z}(1+z))}{(1-ce^{-z})^2} \end{cases} \tag{38}$$

From (30), Noting

$$y \in \left(p, \frac{p}{1-ce^{-q}}\right),\ z \in \left(q, \frac{q}{1-de^{-p}}\right),$$

we get

$$\begin{cases} 1-de^{-y}(1+y) > 1-de^{-p}(1+\frac{q}{1-de^{-p}}) > 0, \\ 1-ce^{-z}(1+z) > 1-ce^{-q}(1+\frac{p}{1-ce^{-q}}) > 0. \end{cases} \tag{39}$$

Therefore from (36) and (37), we obtain

$$f'(y) > 0, g'(z) > 0,$$

for

$$y \in \left(p, \frac{p}{1-ce^{-q}}\right),\ z \in \left(q, \frac{q}{1-de^{-p}}\right).$$

Hence, f, g are increasing functions and this, together with (34) implies that $\lambda_1 = \Lambda_1$. Then, from (34) again, we see that $\lambda_2 = \Lambda_2$. This implies that $\lim_{n\to\infty} y_n = \bar{y}$, $\lim_{n\to\infty} z_n = \bar{z}$. This completes the proof of the theorem. □

Remark 9 *The condition satisfying the unique positive equilibrium of (1.2) is globally asymptotically stable is different from that of [13].*

Theorem 10 *Consider fuzzy difference equation (2) where A, B and the initial values $x_{-i} (i = 0, 1)$, are positive fuzzy numbers, if (19) is satisfied. Then the following statements are true.*
(i) The equation (2) has a unique positive equilibrium.
(ii) The every positive solution x_n of (2) converges to the unique positive equilibrium x with respect to D as $n \to \infty$.

Proof: (i) We consider the system

$$\begin{cases} L_\alpha = A_{l,\alpha} + B_{l,\alpha} L_\alpha e^{-R_\alpha}, \\ R_\alpha = A_{r,\alpha} + B_{r,\alpha} R_\alpha e^{-L_\alpha}. \end{cases} \tag{40}$$

Obviously, system (40) has a unique solution (L_α, R_α).

Let x_n be a positive solution of (2) such that $[x_n]_\alpha = [L_{n,\alpha}, R_{n,\alpha}], \alpha \in (0, 1], n = 0, 1, \cdots$. Then applying Theorem 8 to the system (7) we have

$$\lim_{n\to\infty} L_{n,\alpha} = L_\alpha, \quad \lim_{n\to\infty} R_{n,\alpha} = R_\alpha. \tag{41}$$

From (22) and (41) we have, for $0 < \alpha_1 \le \alpha_2 \le 1$,

$$0 < L_{\alpha_1} \le L_{\alpha_2} \le R_{\alpha_2} \le R_{\alpha_1}. \tag{42}$$

Since $A_{l,\alpha}, A_{r,\alpha}, B_{l,\alpha}, B_{r,\alpha}$ are left continuous, it follows from (40) that L_α, R_α are also left continuous. From (40) and (12) we get

$$\begin{cases} L_\alpha \ge A_{l,\alpha} \ge M_A, \\ R_\alpha = \frac{A_{r,\alpha}}{1-B_{r,\alpha}e^{-L_\alpha}} \le \frac{N_A}{1-N_B e^{-M_A}}. \end{cases} \tag{43}$$

Therefore (43) implies that

$$[L_\alpha, R_\alpha] \subset \left[M_A, \frac{N_A}{1 - N_B e^{-M_A}}\right].$$

From which it is obvious that

$$\begin{cases} \bigcup_{\alpha\in(0,1]}[L_\alpha, R_\alpha] \text{ is compact} \\ \bigcup_{\alpha\in(0,1]}[L_\alpha, R_\alpha] \subset (0, \infty). \end{cases} \quad (44)$$

So from Lemma 2, relations (40), (42), (44) and since $L_\alpha, R_\alpha, \alpha \in (0,1]$, determines a fuzzy number x such that

$$x = A + Bxe^x, \ [x]_\alpha = [L_\alpha, R_\alpha], \ \alpha \in (0,1].$$

and so x is a positive equilibrium of (2).

Provided that there exists another positive equilibrium $\bar{x}$ for (2), then there exist functions $\overline{L_\alpha}$: $(0,1] \to (0,\infty), \overline{R_\alpha} : (0,1] \to (0,\infty)$ such that

$$\bar{x} = A + B\bar{x}e^{\bar{x}}, \ [\bar{x}]_\alpha = [\overline{L_\alpha}, \overline{R_\alpha}], \ \alpha \in (0,1].$$

From which we have

$$\overline{L_\alpha} = A_{l,\alpha} + B_{l,\alpha}\overline{L_\alpha}e^{-\overline{R_\alpha}}, \ \overline{R_\alpha} = A_{r,\alpha} + B_{r,\alpha}\overline{R_\alpha}e^{-\overline{L_\alpha}}.$$

So $L_\alpha = \overline{L_\alpha}, \ R_\alpha = \overline{R_\alpha}, \alpha \in (0,1]$. namely $x = \bar{x}$. This completes the proof of (i).

(ii) From (41) we have

$$\begin{aligned} & \lim_{n\to\infty} D(x_n, x) \\ = & \lim_{n\to\infty} \sup_{\alpha\in(0,1]} \{\max\{|L_{n,\alpha} - L_\alpha|, |R_{n,\alpha} - R_\alpha|\}\} \\ & = 0. \end{aligned} \quad (45)$$

From which it is obvious that every positive solution x_n of Eq.(2) converges the unique equilibrium x with respect to D as $n \to \infty$. □

3 Numerical example

To illustrate our results we give some examples in which the conditions of our propositions hold.

Example 1 Consider the following fuzzy difference equation

$$x_{n+1} = A + Bx_{n-1}e^{-x_n}, \ n = 0, 1, \cdots, \quad (46)$$

where A, B are fuzzy numbers such that

$$A(x) = \begin{cases} 10x - 3, & 0.3 \le x \le 0.4 \\ -5x + 3, & 0.4 \le x \le 0.6 \end{cases} \quad (47)$$

and

$$B(x) = \begin{cases} 10x - 5, & 0.5 \le x \le 0.6 \\ -5x + 4, & 0.6 \le x \le 0.8 \end{cases} \quad (48)$$

We take initial value x_{-1}, x_0 such that

$$x_{-1}(x) = \begin{cases} 10x - 2, & 0.2 \le x \le 0.3 \\ -\frac{10}{3}x + 2, & 0.3 \le x \le 0.6 \end{cases} \quad (49)$$

and

$$x_0(x) = \begin{cases} 5x - 5, & 0.1 \le x \le 0.3 \\ -5x + 2.5, & 0.3 \le x \le 0.5 \end{cases} \quad (50)$$

From (47) and (48), we get, for $\alpha \in (0,1]$,

$$\begin{cases} [A]_\alpha = \left[0.3 + \frac{1}{10}\alpha, 0.6 - \frac{1}{5}\alpha\right], \\ [B]_\alpha = \left[0.5 + \frac{1}{10}\alpha, 0.8 - \frac{1}{5}\alpha\right], \end{cases} \quad (51)$$

And so

$$\overline{\bigcup_{\alpha\in(0,1]} [A]_\alpha} = [0.3, 0.6], \ \overline{\bigcup_{\alpha\in(0,1]} [B]_\alpha} = [0.5, 0.8], \quad (52)$$

Moreover from (49) and (50), we get, for $\alpha \in (0,1]$,

$$\begin{cases} [x_{-1}]_\alpha = \left[0.2 + \frac{1}{10}\alpha, 0.6 - \frac{3}{10}\alpha\right], \\ [x_0]_\alpha = \left[0.1 + \frac{1}{5}\alpha, 0.5 - \frac{1}{5}\alpha\right], \end{cases} \quad (53)$$

It follows that

$$\overline{\bigcup_{\alpha\in(0,1]} [x_{-1}]_\alpha} = [0.2, 0.6], \ \overline{\bigcup_{\alpha\in(0,1]} [x_0]_\alpha} = [0.1, 0.5] \quad (54)$$

From (46), it results in a coupled system of difference equation with parameter $\alpha \in (0,1]$,

$$\begin{cases} L_{n+1,\alpha} = 0.3 + \frac{1}{10}\alpha + \left(0.5 + \frac{1}{10}\alpha\right) L_{n-1,\alpha}e^{-R_{n,\alpha}}, \\ R_{n+1,\alpha} = 0.6 - \frac{1}{5}\alpha + \left(0.8 - \frac{1}{5}\alpha\right) R_{n-1,\alpha}e^{-L_{n,\alpha}}. \end{cases} \quad (55)$$

Therefore, $B_{r,\alpha} < e^{-A_{l,\alpha}}$, for any $\alpha \in (0,1]$, namely, the condition (19) is satisfied, so from Theorem 6, we have that every positive solution x_n of Eq.(46) is bounded and persists. In addition, Eq.(46) has a unique positive equilibrium

$$x = (0.1292, 0.5, 1.8229)$$

Moreover every positive solution x_n of Eq.(46) converges the unique equilibrium x with respect to D as $n \to \infty$. (see Fig.1-Fig.4)

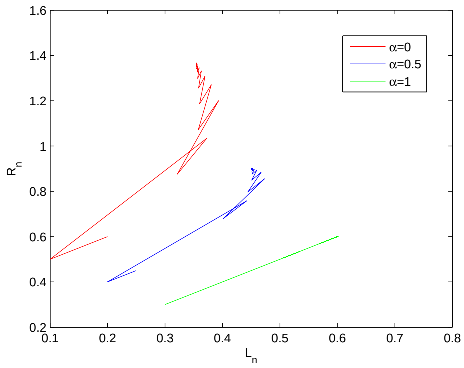

Figure 1: The dynamics of system (55)

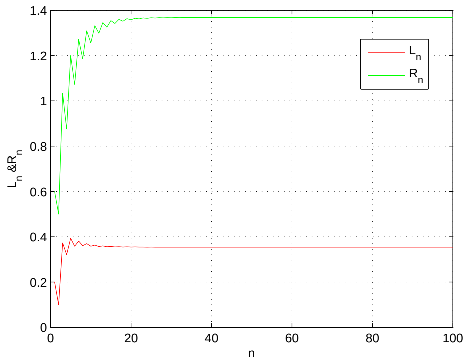

Figure 2: The solution of system (55) at $\alpha = 0$.

4 Conclusion

In this work, fuzzy Logistic difference

$$x_{n+1} = A + Bx_{n-1}e^{-x_n}, n = 0, 1, \cdots,$$

is discussed. Firstly, the existence of positive solution to this equation is proved. Secondly, we find that under condition $B_{r,\alpha} < e^{A_{l,\alpha}}$, the positive solutions of fuzzy Logistic difference are bounded and persistence, and there exists unique positive equilibrium x such that every positive solution converges it. Finally, an example is given to illustrate our results obtained.

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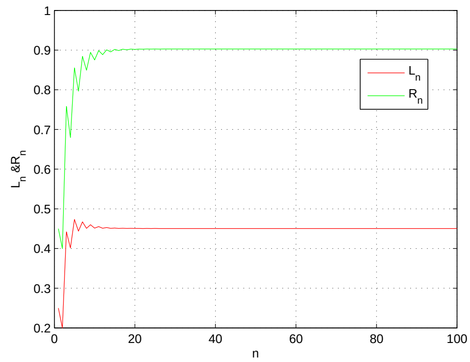

Figure 3: The solution of system (55) at $\alpha = 0.5$.

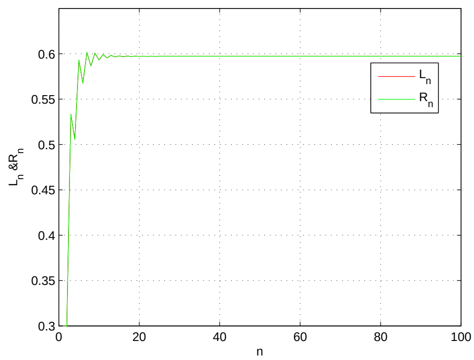

Figure 4: The solution of system (55) at $\alpha = 1$.

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