Dynamic Behaviors of an Almost Periodic Volterra Integro Dynamic Equation on Time Scales

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Abstract: This paper is concerned with an almost periodic Volterra integro dynamic equation on time scales. Based on the theory of calculus on time scales, by using differential inequality theory and constructing a suitable Lyapunov functional, sufficient conditions which guarantee the permanence and the global attractivity of the system are obtained. Then, by using the properties of almost periodic functions and Razumikhin type theorem, sufficient conditions which guarantee the existence of a positive almost periodic solution of the system are obtained. Finally, an example and numerical simulations are presented to illustrate the feasibility and effectiveness of the results.

Key–Words: Permanence; Global attractivity; Almost periodic solution; Time scale.

1 Introduction

In the past few years, many papers have appeared in the literature on Volterra equations on particular time scales such as $\mathbb{R}$ and $\mathbb{Z}$; see, for example, [1-4]. However, in the natural world, there are many species whose developing processes are both continuous and discrete. Hence, using the only differential equation or difference equation can't accurately describe the law of their developments. Therefore, there is a need to establish correspondent dynamic models on new time scales, one may see [5-15].

Recently, Volterra integro dynamic equations on time scales received more researchers' special attention, see, for example, [16,17] and the references therein. However, ecosystem in the real world are continuously disturbed by unpredictable forces which can result in changes in the biological parameters. Hence, if the various constituent components of the temporally nonuniform environment is with incommensurable (nonintegral multiples) periods, then one has to consider the environment to be almost periodic since there is no a priori reason to expect the existence of periodic solutions. Therefore, if we consider the effects of the environmental factors, the assumption of almost periodicity is more realistic, more important and more general.

To the best of the authors' knowledge, there are few papers published on the dynamic behaviors (permanence, global attractivity, almost periodicity, etc.) of Volterra integro dynamic equation on time scales.

Motivated by the above, in the present paper, we shall study an almost periodic Volterra integro dynamic equation on time scales as follows:

$$\begin{aligned} x^{\Delta}(t) &= x(t)[r(t)-a(t)x(t)-b(t)x(\sigma(t)) \\ &\quad -c(t)\int_{-\infty}^{t} e_{-\eta}(t,\sigma(s))x(s)\Delta s], \quad (1) \end{aligned}$$

where $t\in\mathbb{T}$, $\mathbb{T}$ is an almost time scale.

For convenience, we introduce the notation

$$f^u=\sup_{t\in\mathbb{T}} f(t),\ f^l=\inf_{t\in\mathbb{T}} f(t),$$

where f is a positive and bounded function. Throughout this paper, we assume that all the coefficients $r(t),a(t),b(t),c(t),\eta(t)$ of the almost periodic system (1) are continuous, positive almost periodic functions, and

$$\min\{r^l,a^l,b^l,c^l,\eta^l\}>0,$$
$$\max\{r^u,a^u,b^u,c^u,\eta^u\}<+\infty.$$

The initial condition of system (1) in the form

$$x(t_0)=x_0, x_0>0, t_0\in\mathbb{T}. \quad (2)$$

The aim of this paper is, based on the theory of calculus on time scales, by using differential inequality theory and constructing a suitable Lyapunov functional, to obtain sufficient conditions for the permanence and the global attractivity of system (1); by using the properties of almost periodic functions and

Razumikhin type theorem, to obtain sufficient conditions for the existence of a positive almost periodic solution of system (1).

In this paper, for each interval $\mathbb{I}$ of $\mathbb{T}$, we denote by $\mathbb{I}_{\mathbb{T}} = \mathbb{I} \cap \mathbb{T}$.

2 Preliminaries

Let $\mathbb{T}$ be a nonempty closed subset (time scale) of $\mathbb{R}$. The forward and backward jump operators $\sigma, \rho : \mathbb{T} \to \mathbb{T}$ and the graininess $\mu : \mathbb{T} \to \mathbb{R}^+$ are defined, respectively, by

$$\sigma(t) = \inf\{s \in \mathbb{T} : s > t\},$$
$$\rho(t) = \sup\{s \in \mathbb{T} : s < t\},$$
$$\mu(t) = \sigma(t) - t.$$

A point $t \in \mathbb{T}$ is called left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$, left-scattered if $\rho(t) < t$, right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$, and right-scattered if $\sigma(t) > t$. If $\mathbb{T}$ has a left-scattered maximum m, then $\mathbb{T}^k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}^k = \mathbb{T}$. If $\mathbb{T}$ has a right-scattered minimum m, then $\mathbb{T}_k = \mathbb{T}\backslash\{m\}$; otherwise $\mathbb{T}_k = \mathbb{T}$.

A function $f : \mathbb{T} \to \mathbb{R}$ is right-dense continuous provided it is continuous at right-dense point in $\mathbb{T}$ and its left-side limits exist at left-dense points in $\mathbb{T}$. If f is continuous at each right-dense point and each left-dense point, then f is said to be a continuous function on $\mathbb{T}$. The set of continuous functions $f : \mathbb{T} \to \mathbb{R}$ will be denoted by $C(\mathbb{T}) = C(\mathbb{T}, \mathbb{R})$.

For $y : \mathbb{T} \to \mathbb{R}$ and $t \in \mathbb{T}^k$, we define the delta derivative of $y(t)$, $y^\Delta(t)$, to be the number (if it exists) with the property that for a given $\varepsilon > 0$, there exists a neighborhood U of t such that

$$\big|[y(\sigma(t)) - y(s)] - y^\Delta(t)[\sigma(t) - s]\big| < \varepsilon|\sigma(t) - s|$$

for all $s \in U$.

If y is continuous, then y is right-dense continuous, and y is delta differentiable at t, then y is continuous at t.

Let y be right-dense continuous, if $Y^\Delta(t) = y(t)$, then we define the delta integral by

$$\int_a^t y(s)\Delta s = Y(t) - Y(a).$$

The basic theories of calculus on time scales, one can see [18].

A function $p : \mathbb{T} \to \mathbb{R}$ is called regressive provided $1+\mu(t)p(t) \neq 0$ for all $t \in \mathbb{T}^k$. The set of all regressive and rd-continuous functions $p : \mathbb{T} \to \mathbb{R}$ will be denoted by $\mathcal{R} = \mathcal{R}(\mathbb{T}, \mathbb{R})$. Define the set $\mathcal{R}^+ = \mathcal{R}^+(\mathbb{T}, \mathbb{R}) = \{p \in \mathcal{R} : 1 + \mu(t)p(t) > 0, \forall\, t \in \mathbb{T}\}$.

If r is a regressive function, then the generalized exponential function e_r is defined by

$$e_r(t, s) = \exp\left\{\int_s^t \xi_{\mu(\tau)}(r(\tau))\Delta\tau\right\}$$

for all $s, t \in \mathbb{T}$, with the cylinder transformation

$$\xi_h(z) = \begin{cases} \frac{\mathrm{Log}(1+hz)}{h}, & \text{if } h \neq 0, \\ z, & \text{if } h = 0. \end{cases}$$

Let $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, define

$$p\oplus q = p+q+\mu pq, \ \ominus p = -\frac{p}{1+\mu p}, \ p\ominus q = p\oplus(\ominus q).$$

Lemma 1. (see [18]) *If $p, q : \mathbb{T} \to \mathbb{R}$ be two regressive functions, then*
(i) $e_0(t, s) \equiv 1$ and $e_p(t, t) \equiv 1$;
(ii) $e_p(\sigma(t), s) = (1 + \mu(t)p(t))e_p(t, s)$;
(iii) $e_p(t, s) = \frac{1}{e_p(s,t)} = e_{\ominus p}(s, t)$;
(iv) $e_p(t, s)e_p(s, r) = e_p(t, r)$;
(v) $\frac{e_p(t,s)}{e_q(t,s)} = e_{p\ominus q}(t, s)$;
(vi) $(e_p(t, s))^\Delta = p(t)e_p(t, s)$.

Lemma 2. (see [19]) *Assume that $a > 0, b > 0$ and $-a \in \mathcal{R}^+$. Then*

$$y^\Delta(t) \geq (\leq) b - ay(t), \ y(t) > 0, \ t \in [t_0, \infty)_{\mathbb{T}}$$

implies

$$y(t) \geq (\leq)\frac{b}{a}[1+(\frac{ay(t_0)}{b}-1)e_{(-a)}(t, t_0)], \ t \in [t_0, \infty)_{\mathbb{T}}.$$

Lemma 3. (see [19]) *Assume that $a > 0, b > 0$. Then*

$$y^\Delta(t) \leq (\geq)y(t)(b-ay(\sigma(t))), \ y(t) > 0, \ t \in [t_0, \infty)_{\mathbb{T}}$$

implies

$$y(t) \leq (\geq)\frac{b}{a}[1+(\frac{b}{ay(t_0)}-1)e_{\ominus b}(t, t_0)], \ t \in [t_0, \infty)_{\mathbb{T}}.$$

Let $\mathbb{T}$ be a time scale with at least two positive points, one of them being always one: $1 \in \mathbb{T}$, there exists at least one point $t \in \mathbb{T}$ such that $0 < t \neq 1$. Define the natural logarithm function on the time scale $\mathbb{T}$ by

$$L_{\mathbb{T}}(t) = \int_1^t \frac{1}{\tau}\Delta\tau, \ t \in \mathbb{T} \cap (0, +\infty).$$

Lemma 4. (see [20]) *Assume that $x : \mathbb{T} \to \mathbb{R}^+$ is strictly increasing and $\widetilde{\mathbb{T}} := x(\mathbb{T})$ is a time scale. If $x^\Delta(t)$ exists for $t \in \mathbb{T}^k$, then*

$$\frac{\Delta}{\Delta t}L_{\mathbb{T}}(x(t)) = \frac{x^\Delta(t)}{x(t)}.$$

Lemma 5. (see [18]) *Assume that $f, g : \mathbb{T} \to \mathbb{R}$ are differentiable at $t \in \mathbb{T}^k$, then $fg : \mathbb{T} \to \mathbb{R}$ is differentiable at t with*

$$\begin{aligned}(fg)^{\Delta}(t) &= f^{\Delta}(t)g(t) + f(\sigma(t))g^{\Delta}(t)\\ &= f(t)g^{\Delta}(t) + f^{\Delta}(t)g(\sigma(t)).\end{aligned}$$

Lemma 6. (see [18]) *Let $a \in \mathbb{T}^k$, if $f : \mathbb{T} \times \mathbb{T}^k \to \mathbb{R}$ is continuous at (t, t), where $t \in \mathbb{T}^k$ with $t > a$, and $f^{\Delta}(t, \cdot)$ is rd-continuous on $[a, \sigma(t)]_{\mathbb{T}}$, then $g(t) = \int_a^t f(t, s)\Delta s$ is differentiable at t with*

$$g^{\Delta}(t) = \int_a^t f^{\Delta}(t, s)\Delta s + f(\sigma(t), t),$$

where f^{Δ} denotes the derivative of f with respect to the first variable.

Definition 7. (see [21]) *A time scale $\mathbb{T}$ is called an almost periodic time scale if*

$$\Pi = \{\tau \in \mathbb{R} : t \pm \tau \in \mathbb{T}, \forall t \in \mathbb{T}\}.$$

Definition 8. (see [21]) *Let $\mathbb{T}$ be an almost periodic time scale. A function $f : \mathbb{T} \to \mathbb{R}$ is called an almost periodic function if the ε-translation set of f*

$$E\{\varepsilon, f\} = \{\tau \in \Pi : |f(t + \tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}\}$$

is a relatively dense set in $\mathbb{T}$ for all $\varepsilon > 0$; that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$, such that in any interval of length $l(\varepsilon)$, there exists at least a $\tau \in E\{\varepsilon, f\}$ such that

$$|f(t + \tau) - f(t)| < \varepsilon, \forall t \in \mathbb{T}.$$

τ is called the ε-translation number of f and $l(\varepsilon)$ is called the inclusion length of $E\{\varepsilon, f\}$.

The relevant definitions and the properties of almost periodic functions, see [21-24]. Similar to the proof of Corollary 1.2 in [24], we can get the following lemma.

Lemma 9. *Let $\mathbb{T}$ be an almost periodic time scale. If $f(t), g(t)$ are almost periodic functions, then, for any $\varepsilon > 0$, $E\{\varepsilon, f\} \cap E\{\varepsilon, g\}$ is a nonempty relatively dense set in $\mathbb{T}$; that is, for any given $\varepsilon > 0$, there exists a constant $l(\varepsilon) > 0$, such that in any interval of length $l(\varepsilon)$, there exists at least a $\tau \in E\{\varepsilon, f\} \cap E\{\varepsilon, g\}$ such that*

$$|f(t + \tau) - f(t)| < \varepsilon, |g(t + \tau) - g(t)| < \varepsilon, \forall t \in \mathbb{T}.$$

Let $C = C([-\tau, 0]_{\mathbb{T}}, \mathbb{R}^n), H^* \in \mathbb{R}^+$. Denote $C_{H^*} = \{\varphi, \varphi \in C, \|\varphi\| < H^*\}$, $S_{H^*} = \{x, x \in \mathbb{R}^n, \|x\| < H^*\}$, $\|\varphi\| = \sup_{\theta \in [-\tau, 0]_{\mathbb{T}}} |\varphi(\theta)|$.

Consider the system

$$x^{\Delta} = f(t, x), \tag{3}$$

where $f(t, \phi)$ is continuous in $(t, \phi) \in \mathbb{R} \times C$ and almost periodic in t uniformly for $\phi \in C_{H^*}, C_{H^*} \subseteq C$. $\forall \alpha > 0, \exists L(\alpha) > 0$ such that $|f(t, \phi)| \leq L(\alpha)$, as $t \in \mathbb{T}, \phi \in C_{\alpha}$.

In order to investigate the almost periodic solution of system (3), we introduce the associate product system of system (3)

$$x^{\Delta} = f(t, x),\ y^{\Delta} = f(t, y). \tag{4}$$

Lemma 10. (see [25]) *Assume that there exists a Lyapunov function $V(t, x, y)$ defined on $[0, +\infty)_{\mathbb{T}} \times S_{H^*} \times S_{H^*}$, which satisfies the following conditions:*

(1) *$\alpha(|x-y|) \leq V(t, x, y) \leq \beta(|x-y|)$, where $\alpha(s)$ and $\beta(s)$ are continuous, increasing and positive definite;*

(2) *$|V(t, x_1, y_1) - V(t, x_2, y_2)| \leq \omega(|x_1 - x_2| + |y_1 - y_2|)$, where $\omega > 0$ is a constant;*

(3) *$D^+V^{\Delta}_{(4)}(t, x, y) \leq -\lambda V(t, x, y)$, where $\lambda > 0$ is a constant.*

Moreover, assumes that (3) has a solution $\xi(t)$ such that $\|\xi\| \leq H < H^*$ for $t \in [t_0, +\infty)_{\mathbb{T}}$. Then system (3) has a unique almost periodic solution which is uniformly asymptotic stable.

Now, we define a new variable

$$y(t) = \int_{-\infty}^{t} e_{-\eta}(t, \sigma(s))x(s)\Delta s, t \in \mathbb{T}, \tag{5}$$

then by Lemma 1 and Lemma 6, system (1) can be transformed into the following system

$$\begin{cases} x^{\Delta}(t) = x(t)[r(t) - a(t)x(t) - b(t)x(\sigma(t)) \\ \qquad\qquad -c(t)y(t)], \\ y^{\Delta}(t) = -\eta(t)y(t) + x(t). \end{cases} \tag{6}$$

The initial condition of system (6) is

$$x(t_0) = x_0, y(t_0) = y_0, x_0 > 0, y_0 > 0, t_0 \in \mathbb{T}. \tag{7}$$

System (6) and system (1) have the same dynamic behaviors.

3 Permanence and attractivity

Assume that the coefficients of system (6) satisfy

(H_1) $-\eta \in \mathcal{R}^+$;

(H_2) $r^l > a^u M_1 + c^u M_2$.

Theorem 11. *Let $(x(t), y(t))$ be any positive solution of system (6) with initial condition (7). If (H_2) hold, then system (6) is permanent, that is, any positive solution $(x(t), y(t))$ of system (6) satisfies*

$$m_1 \le \liminf_{t\to+\infty} x(t) \le \limsup_{t\to+\infty} x(t) \le M_1, \quad (8)$$

$$m_2 \le \liminf_{t\to+\infty} y(t) \le \limsup_{t\to+\infty} y(t) \le M_2, \quad (9)$$

especially if $m_1 \le x_0 \le M_1$, $m_2 \le y_0 \le M_2$, then

$$m_1 \le x(t) \le M_1,\ m_2 \le y(t) \le M_2,\ t \in [t_0, +\infty)_{\mathbb{T}},$$

where

$$M_1 = \frac{r^u}{b^l},\ M_2 = \frac{M_1}{\eta^l},$$
$$m_1 = \frac{r^l - a^u M_1 - c^u M_2}{b^u},$$
$$m_2 = \frac{m_1}{\eta^u}.$$

Proof. Assume that $(x(t), y(t))$ be any positive solution of system (6) with initial condition (7). From the first equation of system (6), we have

$$x^\Delta(t) \le x(t)(r^u - b^l x(\sigma(t))). \quad (10)$$

By Lemma 3, we can get

$$\limsup_{t\to+\infty} x(t) \le \frac{r^u}{b^l} := M_1.$$

Then, for arbitrarily small positive constant $\varepsilon > 0$, there exists a $T_1 > 0$ such that

$$x(t) < M_1 + \varepsilon,\ \forall t \in [T_1, +\infty]_{\mathbb{T}}.$$

From the second equation of system (6), when $t \in [T_1, +\infty)_{\mathbb{T}}$,

$$y^\Delta(t) < -\eta^l y(t) + (M_1 + \varepsilon).$$

Let $\varepsilon \to 0$, then

$$y^\Delta(t) \le -\eta^l y(t) + M_1. \quad (11)$$

By Lemma 2, we can get

$$\limsup_{t\to+\infty} y(t) = \frac{M_1}{\eta^l} := M_2.$$

Then, for arbitrarily small positive constant $\varepsilon > 0$, there exists a $T_2 > T_1$ such that

$$y(t) < M_2 + \varepsilon,\ \forall t \in [T_2, +\infty]_{\mathbb{T}}.$$

On the other hand, from the first equation of system (6), when $t \in [T_2, +\infty)_{\mathbb{T}}$,

$$x^\Delta(t) > x(t)[r^l - a^u(M_1 + \varepsilon) - b^u x(\sigma(t)) - c^u(M_2 + \varepsilon)].$$

Let $\varepsilon \to 0$, then

$$x^\Delta(t) \ge x(t)[r^l - a^u M_1 - b^u x(\sigma(t)) - c^u M_2]. \quad (12)$$

By Lemma 3, we can get

$$\liminf_{t\to+\infty} x(t) = \frac{r^l - a^u M_1 - c^u M_2}{b^u} := m_1.$$

Then, for arbitrarily small positive constant $\varepsilon > 0$, there exists a $T_3 > T_2$ such that

$$x(t) > m_1 - \varepsilon,\ \forall t \in [T_3, +\infty]_{\mathbb{T}}.$$

From the second equation of system (6), when $t \in [T_3, +\infty)_{\mathbb{T}}$,

$$y^\Delta(t) > -\eta^u y(t) + (m_1 - \varepsilon).$$

Let $\varepsilon \to 0$, then

$$y^\Delta(t) \ge -\eta^u y(t) + m_1. \quad (13)$$

By Lemma 2, we can get

$$\liminf_{t\to+\infty} y(t) = \frac{m_1}{\eta^u} := m_2.$$

Then, for arbitrarily small positive constant $\varepsilon > 0$, there exists a $T_4 > T_3$ such that

$$y(t) > m_2 - \varepsilon,\ \forall t \in [T_4, +\infty]_{\mathbb{T}}.$$

In special case, if $m_1 \le x_0 \le M_1$, $m_2 \le y_0 \le M_2$, by Lemma 2 and Lemma 3, it follows from (10)-(13) that

$$m_1 \le x(t) \le M_1,\ m_2 \le y(t) \le M_2,\ t \in [t_0, +\infty)_{\mathbb{T}},$$

This completes the proof. □

Theorem 12. *In addition to conditions (H_1) and (H_2), assume further that the coefficients of system (6) satisfy the following conditions:*

(H_3) $a^l - 1 > 0$;

(H_4) $\eta^l - c^u > 0$.

Then the solution of system (6) is globally attractive.

Proof. Let $z_1(t) = (x_1(t), y_1(t))$ and $z_2(t) = (x_2(t), y_2(t))$ be any two positive solutions of system (6). It follows from (8)-(9) that for sufficient small positive constant ε_0 $(0 < \varepsilon_0 < \min\{m_1, m_2\})$, there exists a $T > 0$ such that

$$\begin{array}{l} m_1 - \varepsilon_0 < x_i(t) < M_1 + \varepsilon_0, \\ m_2 - \varepsilon_0 < y_i(t) < M_2 + \varepsilon_0, \\ \forall\, t \in [T, +\infty)_{\mathbb{T}}, i = 1, 2. \end{array} \qquad (14)$$

Since $x_i(t), i = 1, 2$ are positive, bounded and differentiable functions on $\mathbb{T}$, then there exists a positive, bounded and differentiable function $m(t), t \in \mathbb{T}$, such that $x_i(t)(1+m(t)), i = 1, 2$ are strictly increasing on $\mathbb{T}$. By Lemma 4 and Lemma 5, we have

$$\begin{aligned} & \frac{\Delta}{\Delta t} L_{\mathbb{T}}(x_i(t)[1 + m(t)]) \\ = & \frac{x_i^{\Delta}(t)[1 + m(t)] + x_i(\sigma(t))m^{\Delta}(t)}{x_i(t)[1 + m(t)]} \\ = & \frac{x_i^{\Delta}(t)}{x_i(t)} + \frac{x_i(\sigma(t))m^{\Delta}(t)}{x_i(t)[1 + m(t)]},\ i = 1, 2. \end{aligned}$$

Here, we can choose a function $m(t)$ such that $\frac{|m^{\Delta}(t)|}{1+m(t)}$ is bounded on $\mathbb{T}$, that is, there exist two positive constants $\zeta > 0$ and $\xi > 0$ such that $0 < \zeta < \frac{|m^{\Delta}(t)|}{1+m(t)} < \xi$, $\forall t \in \mathbb{T}$.

Set

$$\begin{aligned} V(t) = & |e_{-\delta}(t, T)|(|L_{\mathbb{T}}(x_1(t)(1 + m(t))) \\ & -L_{\mathbb{T}}(x_2(t)(1 + m(t)))| \\ & +|y_1(t) - y_2(t)|), \end{aligned}$$

where $\delta \geq 0$ is a constant (if $\mu(t) = 0$, then $\delta = 0$; if $\mu(t) > 0$, then $\delta > 0$). It follows from the mean value theorem of differential calculus on time scales for $t \in [T, +\infty)_{\mathbb{T}}$,

$$\begin{aligned} & \frac{1}{M_1 + \varepsilon_0}|x_1(t) - x_2(t)| \\ \leq & |L_{\mathbb{T}}(x_1(t)(1 + m(t))) - L_{\mathbb{T}}(x_2(t)(1 + m(t)))| \\ \leq & \frac{1}{m_1 - \varepsilon_0}|x_1(t) - x_2(t)|. \end{aligned} \qquad (15)$$

Let $\gamma = \min\{(m_1 - \varepsilon_0)(a^l - 1), \eta^l - c^u\}$. We divide the proof into two cases.

Case I. If $\mu(t) > 0$, set $\delta > \max\{(b^u + \frac{\xi}{m_1})M_1, \gamma\}$ and $1 - \mu(t)\delta < 0$. Calculating the upper right derivatives of $V(t)$ along the solution of system (6), it follows from (14), (15), (H_3) and (H_4) that for $t \in [T, +\infty)_{\mathbb{T}}$,

$$\begin{aligned} & D^+V^{\Delta}(t) \\ = & |e_{-\delta}(t, T)|\mathrm{sgn}(x_1(t) - x_2(t))\Bigg[\frac{x_1^{\Delta}(t)}{x_1(t)} - \frac{x_2^{\Delta}(t)}{x_2(t)} \\ & +\frac{m^{\Delta}(t)}{1 + m(t)}\left(\frac{x_1(\sigma(t))}{x_1(t)} - \frac{x_2(\sigma(t))}{x_2(t)}\right)\Bigg] \\ & -\delta|e_{-\delta}(t, T)||L_{\mathbb{T}}(x_1(\sigma(t))(1 + m(\sigma(t)))) \\ & -L_{\mathbb{T}}(x_2(\sigma(t))(1 + m(\sigma(t))))| \\ & +|e_{-\delta}(t, T)|\mathrm{sgn}(y_1(t) - y_2(t))(y_1^{\Delta}(t) - y_2^{\Delta}(t)) \\ & -\delta|e_{-\delta}(t, T)||y_1(\sigma(t)) - y_2(\sigma(t))| \\ = & |e_{-\delta}(t, T)|\mathrm{sgn}(x_1(t) - x_2(t)) \\ & \times\Bigg[- a(t)(x_1(t) - x_2(t)) \\ & -b(t)(x_1(\sigma(t)) - x_2(\sigma(t))) \\ & -c(t)(y_1(t) - y_2(t))] \\ & +\frac{m^{\Delta}(t)}{1 + m(t)}\frac{x_1(\sigma(t))x_2(t) - x_1(t)x_2(\sigma(t))}{x_1(t)x_2(t)}\Bigg] \\ & -\delta|e_{-\delta}(t, T)||L_{\mathbb{T}}(x_1(\sigma(t))(1 + m(\sigma(t)))) \\ & -L_{\mathbb{T}}(x_2(\sigma(t))(1 + m(\sigma(t))))| \\ & +|e_{-\delta}(t, T)|\mathrm{sgn}(y_1(t) - y_2(t)) \\ & \times[-\eta(t)(y_1(t) - y_2(t)) + (x_1(t) - x_2(t))] \\ & -\delta|e_{-\delta}(t, T)||y_1(\sigma(t)) - y_2(\sigma(t))| \\ = & |e_{-\delta}(t, T)|\mathrm{sgn}(x_1(t) - x_2(t)) \\ & \times\Bigg[- a(t)(x_1(t) - x_2(t)) \\ & -b(t)(x_1(\sigma(t)) - x_2(\sigma(t))) \\ & -c(t)(y_1(t) - y_2(t))] \\ & +\frac{m^{\Delta}(t)}{1 + m(t)}\Bigg(\frac{x_1(\sigma(t))(x_2(t) - x_1(t))}{x_1(t)x_2(t)} \\ & +\frac{x_1(t)(x_1(\sigma(t)) - x_2(\sigma(t)))}{x_1(t)x_2(t)}\Bigg)\Bigg] \\ & -\delta|e_{-\delta}(t, T)||L_{\mathbb{T}}(x_1(\sigma(t))(1 + m(\sigma(t)))) \\ & -L_{\mathbb{T}}(x_2(\sigma(t))(1 + m(\sigma(t))))| \\ & +|e_{-\delta}(t, T)|\mathrm{sgn}(y_1(t) - y_2(t)) \\ & \times[-\eta(t)(y_1(t) - y_2(t)) + (x_1(t) - x_2(t))] \\ & -\delta|e_{-\delta}(t, T)||y_1(\sigma(t)) - y_2(\sigma(t))| \\ \leq & -|e_{-\delta}(t, T)|\left[a(t) - 1 + \frac{|m^{\Delta}(t)|}{1 + m(t)}\frac{x_1(\sigma(t))}{x_1(t)x_2(t)}\right] \\ & \times|x_1(t) - x_2(t)| \\ & -|e_{-\delta}(t, T)|\Bigg[\frac{\delta}{M_1 + \varepsilon_0} - b(t) \\ & -\frac{|m^{\Delta}(t)|}{1 + m(t)}\frac{1}{x_2(t)}\Bigg]|x_1(\sigma(t)) - x_2(\sigma(t))| \\ & -|e_{-\delta}(t, T)|(\eta(t) - c(t))|y_1(t) - y_2(t)| \end{aligned}$$

$$\begin{aligned}
&-\delta|e_{-\delta}(t,T)||y_1(\sigma(t))-y_2(\sigma(t))|\\
\le\;& -|e_{-\delta}(t,T)|(a^l-1)|x_1(t)-x_2(t)|\\
&-|e_{-\delta}(t,T)|(\eta^l-c^u)|y_1(t)-y_2(t)|\\
\le\;& -|e_{-\delta}(t,T)|((m_1-\varepsilon_0)(a^l-1)\\
&\times|L_{\mathbb{T}}(x_1(t)(1+m(t)))\\
&-L_{\mathbb{T}}(x_2(t)(1+m(t)))|\\
&+(\eta^l-c^u)|y_1(t)-y_2(t)|)\\
\le\;& -\gamma|e_{-\delta}(t,T)|(|L_{\mathbb{T}}(x_1(t)(1+m(t)))\\
&-L_{\mathbb{T}}(x_2(t)(1+m(t)))|+|y_1(t)-y_2(t)|)\\
=\;& -\gamma V(t). \qquad (16)
\end{aligned}$$

By the comparison theorem and (16), we have

$$\begin{aligned}
&V(t)\le|e_{-\gamma}(t,T)|V(T)\\
&<2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\bigg)|e_{-\gamma}(t,T)|,
\end{aligned}$$

that is,

$$\begin{aligned}
&|e_{-\delta}(t,T)|(|L_{\mathbb{T}}(x_1(t)(1+m(t)))\\
&-L_{\mathbb{T}}(x_2(t)(1+m(t)))|+|y_1(t)-y_2(t)|)\\
&<2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\bigg)|e_{-\gamma}(t,T)|,
\end{aligned}$$

then

$$\begin{aligned}
&\frac{1}{M_1+\varepsilon_0}|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)|\\
&<2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\bigg)\\
&\times|e_{(-\gamma)\ominus(-\delta)}(t,T)|. \qquad (17)
\end{aligned}$$

Since $1-\mu(t)\delta<0$ and $0<\gamma<\delta$, then $(-\gamma)\ominus(-\delta)<0$. It follows from (17) that

$$\lim_{t\to+\infty}|x_1(t)-x_2(t)|=0,\ \lim_{t\to+\infty}|y_1(t)-y_2(t)|=0.$$

Case II. If $\mu(t)=0$, set $\delta=0$, then $\sigma(t)=t$ and $|e_{-\delta}(t,T)|=1$. Calculating the upper right derivatives of $V(t)$ along the solution of system (6), it follows from (14), (15), (H_3) and (H_4) that for $t\in[T,+\infty)_{\mathbb{T}}$,

$$\begin{aligned}
&D^+V^{\Delta}(t)\\
=\;& \mathrm{sgn}(x_1(t)-x_2(t))\bigg(\frac{x_1^{\Delta}(t)}{x_1(t)}-\frac{x_2^{\Delta}(t)}{x_2(t)}\bigg)\\
&+\mathrm{sgn}(y_1(t)-y_2(t))(y_1^{\Delta}(t)-y_2^{\Delta}(t))\\
=\;& \mathrm{sgn}(x_1(t)-x_2(t))[-(a(t)+b(t))\\
&\times(x_1(t)-x_2(t))-c(t)(y_1(t)-y_2(t))]\\
&+\mathrm{sgn}(y_1(t)-y_2(t))[-\eta(t)(y_1(t)-y_2(t))\\
&+(x_1(t)-x_2(t))]\\
\le\;& -(a(t)+b(t)-1)|x_1(t)-x_2(t)|\\
&-(\eta(t)-c(t))|y_1(t)-y_2(t)|\\
\le\;& -((m_1-\varepsilon_0)(a^l+b^l-1)\\
&\times|L_{\mathbb{T}}(x_1(t)(1+m(t)))\\
&-L_{\mathbb{T}}(x_2(t)(1+m(t)))|\\
&+(\eta^l-c^u)|y_1(t)-y_2(t)|)\\
\le\;& -\widehat{\gamma}(|L_{\mathbb{T}}(x_1(t)(1+m(t)))\\
&-L_{\mathbb{T}}(x_2(t)(1+m(t)))|+|y_1(t)-y_2(t)|)\\
\le\;& -\gamma V(t), \qquad (18)
\end{aligned}$$

where $\widehat{\gamma}=\min\{(m_1-\varepsilon_0)(a^l+b^l-1),\eta^l-c^u\}$. By the comparison theorem and (18), we have

$$\begin{aligned}
V(t)\;&\le\;|e_{-\gamma}(t,T)|V(T)\\
&<\;2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\bigg)|e_{-\gamma}(t,T)|,
\end{aligned}$$

that is,

$$\begin{aligned}
&|L_{\mathbb{T}}(x_1(t)(1+m(t)))-L_{\mathbb{T}}(x_2(t)(1+m(t)))|\\
&+|y_1(t)-y_2(t)|\\
&<2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\bigg)|e_{-\gamma}(t,T)|,
\end{aligned}$$

then

$$\begin{aligned}
&\frac{1}{M_1+\varepsilon_0}|x_1(t)-x_2(t)|+|y_1(t)-y_2(t)|\\
&<2\bigg(\frac{M_1+\varepsilon_0}{m_1-\varepsilon_0}+M_2+\varepsilon_0\bigg)|e_{-\gamma}(t,T)|. \quad (19)
\end{aligned}$$

It follows from (19) that

$$\lim_{t\to+\infty}|x_1(t)-x_2(t)|=0,\ \lim_{t\to+\infty}|y_1(t)-y_2(t)|=0.$$

From the above discussion, we can see that the solution of system (6) is globally attractive. This completes the proof. □

4 Almost periodic solution

Let $S(\mathbb{T})$ be the set of all solutions $(x(t),y(t))$ of system (6) satisfying $m_1\le x(t)\le M_1,\ m_2\le y(t)\le M_2$ for all $t\in\mathbb{T}$.

Lemma 13. *$S(\mathbb{T})\neq\emptyset$.*

Proof. By Theorem 11, we see that for any $t_0\in\mathbb{T}$ with $m_1\le x_0\le M_1$, $m_2\le y_0\le M_2$, system (6) has a solution $(x(t),y(t))$ satisfying $m_1\le x(t)\le$

$M_1,\ m_2 \le y(t) \le M_2, t \in [t_0, +\infty)_{\mathbb{T}}$. Since $r(t)$, $a(t)$, $b(t)$, $c(t)$, $\eta(t)$, $\sigma(t)$ are almost periodic, by Lemma 9, there exists a sequence $\{t_n\}$, $t_n \to +\infty$ as $n \to +\infty$ such that $r(t+t_n) \to r(t), a(t+t_n) \to a(t), b(t+t_n) \to b(t), c(t+t_n) \to c(t), \eta(t+t_n) \to \eta(t), \sigma(t+t_n) \to \sigma(t)$ as $n \to +\infty$ uniformly on $\mathbb{T}$.

We claim that $\{x(t+t_n)\}$ and $\{y(t+t_n)\}$ are uniformly bounded and equi-continuous on any bounded interval in $\mathbb{T}$.

In fact, for any bounded interval $[\alpha, \beta]_{\mathbb{T}} \subset \mathbb{T}$, when n is large enough, $\alpha + t_n > t_0$, then $t + t_n > t_0, \forall t \in [\alpha, \beta]_{\mathbb{T}}$. So, $m_1 \le x(t+t_n) \le M_1,\ m_2 \le y(t+t_n) \le M_2$ for any $t \in [\alpha, \beta]_{\mathbb{T}}$, that is, $\{x(t+t_n)\}$ and $\{y(t+t_n)\}$ are uniformly bounded. On the other hand, $\forall t_1, t_2 \in [\alpha, \beta]_{\mathbb{T}}$, from the mean value theorem of differential calculus on time scales, we have

$$\begin{aligned} & |x(t_1+t_n) - x(t_2+t_n)| \\ \le\ & M_1[r^u + (a^u + b^u)M_1 + c^u M_2] \\ & \times |t_1 - t_2|, \end{aligned} \tag{20}$$

$$\begin{aligned} & |y(t_1+t_n) - y(t_2+t_n)| \\ \le\ & (\eta^u M_2 + M_1)|t_1 - t_2|. \end{aligned} \tag{21}$$

The inequalities (20) and (21) show that $\{x(t+t_n)\}$ and $\{y(t+t_n)\}$ are equi-continuous on $[\alpha, \beta]_{\mathbb{T}}$. By the arbitrary of $[\alpha, \beta]_{\mathbb{T}}$, the conclusion is valid.

By Ascoli-Arzela theorem, there exists a subsequence of $\{t_n\}$, we still denote it as $\{t_n\}$, such that

$$x(t+t_n) \to p(t), y(t+t_n) \to q(t),$$

as $n \to +\infty$ uniformly in t on any bounded interval in $\mathbb{T}$. For any $\theta \in \mathbb{T}$, we can assume that $t_n + \theta \ge t_0$ for all n, and let $t \ge 0$, integrate both equations of system (6) from $t_n + \theta$ to $t + t_n + \theta$, we have

$$\begin{aligned} & x(t+t_n+\theta) - x(t_n+\theta) \\ =\ & \int_{t_n+\theta}^{t+t_n+\theta} x(s)[r(s) - a(s)x(s) \\ & -b(s)x(\sigma(s)) - c(s)y(s)]\Delta s \\ =\ & \int_{\theta}^{t+\theta} x(s+t_n)[r(s+t_n) \\ & -a(s+t_n)x(s+t_n) \\ & -b(s+t_n)x(\sigma(s+t_n)) \\ & -c(s+t_n)y(s+t_n)]\Delta s, \end{aligned}$$

and

$$\begin{aligned} & y(t+t_n+\theta) - y(t_n+\theta) \\ =\ & \int_{t_n+\theta}^{t+t_n+\theta} [-\eta(s)y(s) + x(s)]\Delta s \end{aligned}$$

$$= \int_{\theta}^{t+\theta} [-\eta(s+t_n)y(s+t_n) + x(s+t_n)]\Delta s.$$

Using the Lebesgues dominated convergence theorem, we have

$$\begin{aligned} p(t+\theta) - p(\theta) &= \int_{\theta}^{t+\theta} x(s)[r(s) - a(s)x(s) \\ & \quad -b(s)x(\sigma(s)) - c(s)y(s)]\Delta s, \\ q(t+\theta) - q(\theta) &= \int_{\theta}^{t+\theta} [-\eta(s)y(s) + x(s)]\Delta s. \end{aligned}$$

This means that $(p(t), q(t))$ is a solution of system (6), and by the arbitrary of θ, $(p(t), q(t))$ is a solution of system (6) on $\mathbb{T}$. It is clear that

$$m_1 \le p(t) \le M_1,\ m_2 \le q(t) \le M_2,\ \forall t \in \mathbb{T}.$$

This completes the proof. □

Theorem 14. *Assume that the conditions* $(H_1)-(H_4)$ *hold, then system (6) has a unique positive almost periodic solution which is globally attractive.*

Proof. Consider the associated product system of system (6),

$$\begin{cases} x_1^{\Delta}(t) = x_1(t)[r(t) - a(t)x_1(t) \\ \qquad -b(t)x_1(\sigma(t)) - c(t)y_1(t)], \\ y_1^{\Delta}(t) = -\eta(t)y_1(t) + x_1(t), \\ x_2^{\Delta}(t) = x_2(t)[r(t) - a(t)x_2(t) \\ \qquad -b(t)x_2(\sigma(t)) - c(t)y_2(t)], \\ y_2^{\Delta}(t) = -\eta(t)y_2(t) + x_2(t). \end{cases} \tag{22}$$

Let $z(t) = (z_1(t), z_2(t))$ be a positive solution of product system (22), where

$$z_1(t) = (x_1(t), y_1(t)), z_2(t) = (x_2(t), y_2(t)).$$

By using the same Lyapunov functional in Section 3. Set

$$\begin{aligned} & V(t, z_1(t), z_2(t)) \\ =\ & |e_{-\delta}(t, T)|(|L_{\mathbb{T}}(x_1(t)(1+m(t))) \\ & -L_{\mathbb{T}}(x_2(t)(1+m(t)))| \\ & +|y_1(t) - y_2(t)|). \end{aligned}$$

It follows from (15) that

$$\begin{aligned} & \min\{\frac{1}{M_1+\varepsilon_0}, 1\}|e_{-\delta}(t, T)|(|x_1(t) - x_2(t)| \\ & +|y_1(t) - y_2(t)|) \\ \le\ & V(t, z_1(t), z_2(t)) \\ \le\ & \max\{\frac{1}{m_1-\varepsilon_0}, 1\}|e_{-\delta}(t, T)|(|x_1(t) - x_2(t)| \\ & +|y_1(t) - y_2(t)|), \end{aligned}$$

then

$$\begin{aligned}
&\min\{\frac{1}{M_1+\varepsilon_0},1\}|e_{-\delta}(t,T)|(|z_1(t)-z_2(t)|)\\
\leq\ & V(t,z_1(t),z_2(t))\\
\leq\ & \max\{\frac{1}{m_1-\varepsilon_0},1\}|e_{-\delta}(t,T)|(|z_1(t)-z_2(t)|).
\end{aligned}$$

Therefore, condition (1) in Lemma 10 is satisfied.

Since

$$\begin{aligned}
&|V(t,z_1(t),z_2(t))-V(t,\widetilde{z}_1(t),\widetilde{z}_2(t))|\\
=\ & |e_{-\delta}(t,T)|\big||L_{\mathbb{T}}(x_1(t)(1+m(t)))\\
&-L_{\mathbb{T}}(x_2(t)(1+m(t)))|+|y_1(t)-y_2(t)|\\
&-|L_{\mathbb{T}}(\widetilde{x}_1(t)(1+m(t)))\\
&-L_{\mathbb{T}}(\widetilde{x}_2(t)(1+m(t)))|-|\widetilde{y}_1(t)-\widetilde{y}_2(t)|\big|\\
\leq\ & |L_{\mathbb{T}}(x_1(t)(1+m(t)))\\
&-L_{\mathbb{T}}(\widetilde{x}_1(t)(1+m(t)))|+|y_1(t)-\widetilde{y}_1(t)|\\
&+|L_{\mathbb{T}}(x_2(t)(1+m(t)))\\
&-L_{\mathbb{T}}(\widetilde{x}_2(t)(1+m(t)))|+|y_2(t)-\widetilde{y}_2(t)|\\
\leq\ & \max\{\frac{1}{m_1-\varepsilon_0},1\}(|x_1(t)-\widetilde{x}_1(t)|\\
&+|y_1(t)-\widetilde{y}_1(t)|\\
&+|x_2(t)-\widetilde{x}_2(t)|+|y_2(t)-\widetilde{y}_2(t)|)\\
=\ & \max\{\frac{1}{m_1-\varepsilon_0},1\}(|z_1(t)-\widetilde{z}_1(t)|\\
&+|z_2(t)-\widetilde{z}_2(t)|).
\end{aligned}$$

Therefore, condition (2) in Lemma 10 holds.

Next, we shall prove condition (3) in Lemma 10 holds. By the proof of Theorem 12. Calculating the upper right derivatives of $V(t,z_1(t),z_2(t))$ along the solution of system (22), it follows from (16) and (18) that for $t\in[T,+\infty)_{\mathbb{T}}$,

$$D^+V^{\Delta}(t,z_1(t),z_2(t))\leq-\gamma V(t,z_1(t),z_2(t)).$$

Therefore, condition (3) in Lemma 10 holds.

From the above discussion, we can see that all conditions in Lemma 10 hold. By Lemma 10 and Lemma 13, system (6) has a unique almost periodic solution which is uniformly asymptotic stable. Together with Theorem 11 and Theorem 12 that system (6) has a unique positive almost periodic solution which is globally attractive. This completes the proof. □

Remark 15. *Since system (1) and system (6) have the same dynamic behaviors, then under conditions (H_1)-(H_2), system (1) is permanent; under conditions (H_1)-(H_4), system (1) has a unique positive almost periodic solution which is globally attractive.*

5 Example and simulations

Consider the following system on time scales

$$\begin{aligned}
&x^{\Delta}(t)\\
=\ & x(t)[2.75+0.25\sin\sqrt{2}t\\
&-(1.75+0.25\sin t)x(t)-5x(\sigma(t)) \qquad (23)\\
&-0.2\int_{-\infty}^{t}e_{-(0.8+0.1\cos\sqrt{3}t)}(t,\sigma(s))x(s)\Delta s].
\end{aligned}$$

Then, system (23) can be transformed into the following system

$$\begin{cases}
x^{\Delta}(t) = x(t)[2.75+0.25\sin\sqrt{2}t\\
\qquad -(1.75+0.25\sin t)x(t)\\
\qquad -5x(\sigma(t))-0.2y(t)],\\
y^{\Delta}(t) = -(0.8+0.1\cos\sqrt{3}t)y(t)+x(t).
\end{cases}$$

By a direct calculation, we can get

$$\begin{aligned}
&r^u=3, r^l=2.5, a^u=2, a^l=1.5,\\
&b^u=b^l=5, c^u=c^l=0.2,\\
&\eta^u=0.9, \eta^l=0.7,\\
&M_1=0.6000,\ M_2=0.8571,\\
&a^uM_1+c^uM_2=1.0286.
\end{aligned}$$

Obviously,

$$-\eta\in\mathcal{R}^+,$$

$$r^l-(a^uM_1+c^uM_2)=1.4714>0,$$

$$a^l-1=0.5>0,$$

$$\eta^l-c^u=0.5>0,$$

that is, the conditions $(H_1)-(H_4)$ hold. According to Theorem 14 and Remark 15, system (23) has a unique globally attractive positive almost periodic solution.

Dynamic simulations of system (23) with $\mathbb{T}=\mathbb{R}$ and $\mathbb{T}=\mathbb{Z}$, see Figures 1 and 2, respectively. From the Figures 1 and 2, we can see that $x(t)$ is globally attractive.

6 Conclusion

This paper is concerned with an almost periodic Volterra integro dynamic equation on time scales. Firstly, we introduced a new variable, based on the theory of calculus on time scales, the Volterra integro dynamic equation was transformed into a nonlinear differential equations, then, by using differential inequality theory and constructing a suitable Lyapunov functional, sufficient conditions which guarantee the permanence and the global attractivity of the system

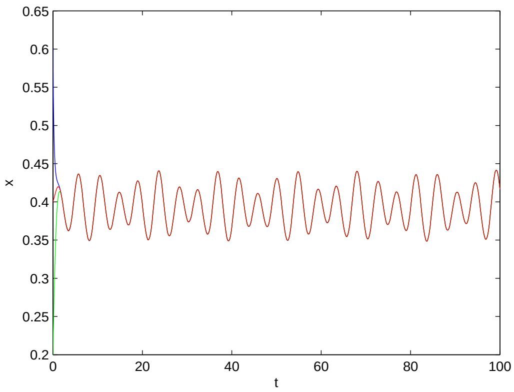

Figure 1: $\mathbb{T} = \mathbb{R}$. Dynamics behavior of system (23) with $x(0) = \{0.2; 0.4; 0.6\}$.

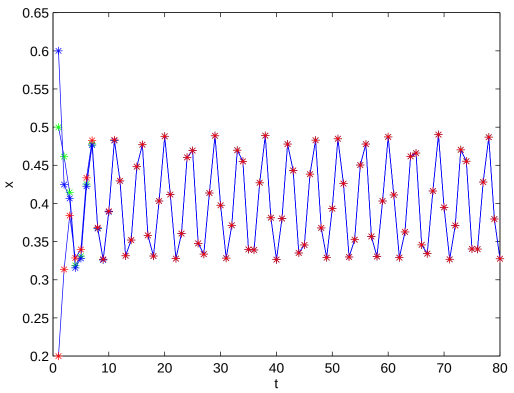

Figure 2: $\mathbb{T} = \mathbb{Z}$. Dynamics behavior of system (23) with $x(0) = \{0.2; 0.5; 0.6\}$.

are obtained; by using the properties of almost periodic functions and Razumikhin type theorem, sufficient conditions which guarantee the existence of a positive almost periodic solution of the system are obtained.

The results obtained in this paper can be applied to the analysis of many other periodic and almost periodic dynamical systems, one may consider the systems which have been studied in [26-30].

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