Products of Volterra-type Operators and Composition Operators on logarithmic Bloch space

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Abstract: Let $D = \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbf{C}$, φ be an analytic self-map of D, and $g : D \longrightarrow \mathbf{C}$ is an analytic map. We characterize the boundedness and compactness of the products of Volterra-type operators and composition operators $C_\varphi U_g$ and $U_g C_\varphi$ on the logarithmic Bloch space $\mathcal{LB}$ and the little logarithmic space $\mathcal{LB}_0$ over the unit disk. Some necessary and sufficient conditions are given for which $C_\varphi U_g$ or $U_g C_\varphi$ is a bounded or a compact operator on $\mathcal{LB}$, or $\mathcal{LB}_0$, respectively. The results extend the known results about the composition operator to the logarithmic Bloch space $\mathcal{LB}$.

Key–Words: Volterra-type operators, Composition operators, Bloch-type spaces, Analytic functions, Boundedness, Compactness

1 Introduction

Let $D = \{z : |z| < 1\}$ be the unit disk in the complex plane $\mathbf{C}$, and $H(D)$ denote the set of all analytic functions on D. An analytic function $f \in H(D)$ is said to belong to the logarithmic Bloch space $\mathcal{LB}$ if

$$\begin{aligned}\|f\|_{\mathcal{LB}} &= \sup_{z\in D}\{(1-|z|^2)\ln(\frac{2}{1-|z|})|f'(z)|\} \\ &< \infty,\end{aligned}$$

and to the little logarithmic Bloch space $\mathcal{LB}_0$ if

$$\lim_{|z|\to 1^-}(1-|z|)\ln\Big(\frac{2}{1-|z|}\Big)|f'(z)| = 0.$$

It can be easily proved that $\mathcal{LB}$ is a Banach space under the norm

$$\|f\|_{\mathcal{L}} = |f(0)| + \|f\|_{\mathcal{LB}}$$

and that $\mathcal{LB}_0$ is a closed subspace of $\mathcal{LB}$. Some basic results about the logarithmic Bloch functions we refer to the references [18, 19, 21, 23] and [25].

Let φ be an analytic self-map on the unit disk D. Associated with φ the composition operator C_φ is defined by

$$C_\varphi f = f \circ \varphi, \qquad f \in H(D).$$

It is interesting to provide a function theoretic characterization when φ induces a bounded or compact composition operator on various function spaces. Boundedness and compactness of composition operators on various function spaces were studied by many authors (see [5, 6, 7, 8, 22, 24]). The author and Yoneda in [18, 25] studied the pointwise multiplier and the composition operator in $\mathcal{LB}$ space respectively.

Suppose that $g : D \longrightarrow \mathbf{C}$ is an analytic map. Let U_g and V_g denote the Volterra-type operators with the analytic symbol g on D respectively:

$$U_g f(z) = \int_0^z f(w)g'(w)\,dw$$

and

$$V_g f(z) = \int_0^z f'(w)g(w)\,dw, \qquad z \in D.$$

At the same time, M_g is the pointwise multiplication determined by

$$M_g(f)(z) = f(z)g(z) = f(0)g(0) + U_g f(z) + V_g f(z).$$

When $g(z) = z$ or $g(z) = \ln(\frac{1}{1-z})$, U_g is the integral operator or the Cesáro operator respectively. These operators U_g, V_g, and M_g are characterized on Q^p spaces by Xiao in [17].

In [9] Pommerenke introduced the Volterra-type operator U_g and showed that U_g is a bounded operator on the Hardy space H^2 if and only if $g \in BMOA$. Brown and Shields in [3] proved that M_g is bounded on the classical Bloch space β_1 if and only if $g \in \mathcal{LB} \bigcap H^\infty$. In [19] the author studied the boundedness and compactness of U_g between the α-Bloch spaces

β_α and the logarithmic Bloch space $\mathcal{LB}$. Boundedness and compactness of U_g acting on various function spaces have been studied in many literature. See [1, 2, 11, 13, 14, 15, 16] for more information.

Here, we consider the products of Volterra-type operators and composition operators, which are defined by

$$(C_\varphi U_g f)(z) = \int_0^{\varphi(z)} f(\zeta)g'(\zeta)\,d\zeta,$$

$$(U_g C_\varphi f)(z) = \int_0^{z} (f\circ\varphi)(\zeta)g'(\zeta)\,d\zeta,\ f\in H(D)$$

and

$$(C_\varphi V_g f)(z) = \int_0^{\varphi(z)} f'(\zeta)g(\zeta)\,d\zeta,$$

$$(V_g C_\varphi f)(z) = \int_0^{z} (f\circ\varphi)'(\zeta)g(\zeta)\,d\zeta,\ f\in H(D).$$

In [4], Li and Stević studied these operators from H^∞ and Bloch spaces to Zygmund spaces. In this paper the boundedness and compactness of these operators in $\mathcal{LB}$ and $\mathcal{LB}_0$ are discussed. As consequences we obtain the boundedness and compactness for U_g and V_g in $\mathcal{LB}$ and $\mathcal{LB}_0$ spaces. These results are new even for a single operator. In what follows C will be used to stand for positive constants which does not depend on the functions but possibly different in different formula.

2 Auxiliary results

In this section, we recall some lemmas, which will be used in the proof of main results of this paper. The first four lemmas may be found in [18].

Lemma 1 *Suppose $f\in\mathcal{LB}$, then*

(i) $|f(z)| \le (2+\ln(\ln\frac{2}{1-|z|}))\|f\|_{\mathcal{L}}$*;*

(ii) $|f(z)| \le 2\ln(\ln\frac{2}{1-|z|})\|f\|_{\mathcal{L}}$*, for* $|z|\ge r_* = 1-2e^{-e^2}$*;*

(iii) $|f(z)-f(tz)| \le \ln(\frac{\ln\frac{2}{1-|z|}}{\ln\frac{2}{1-|tz|}})\|f\|_{\mathcal{LB}}$*, where* $0\le t<1$.

Lemma 2 *If $f\in\mathcal{LB}_0$, then*

$$\lim_{|z|\to1^-}\frac{|f(z)|}{\ln(\ln\frac{2}{1-|z|})}=0.$$

Lemma 3 *Let* $f(z)=\frac{(1-|z|)\ln\frac{2}{1-|z|}}{|1-z|\ln\frac{4}{|1-z|}}$*,* $z\in D$*. Then* $|f(z)|<2$.

Lemma 4 *Let* $g(x)=(1-x)\ln\dfrac{2}{1-x}$*,* $x\in[0,1)$*. Then* $\dfrac{g(x)}{g(tx)}\le 2$ *for each* $t\in[0,1]$.

Lemma 5 *Suppose* $f\in\mathcal{LB}$*, then* $\|f_t\|_{\mathcal{L}}\le 4\|f\|_{\mathcal{L}}$*,* $0<t<1$*, where* $f_t(z)=f(tz)$.

It can be easily proved by applying Lemma 4.

Lemma 6 *Let g be an analytic function on the unit disc D and φ an analytic self-map of D. If $C_\varphi U_g$(or U_gC_φ, $C_\varphi V_g$, V_gC_φ) is a bounded operator in the logarithmic little Bloch space $\mathcal{LB}_0$, then $C_\varphi U_g$(or U_gC_φ, $C_\varphi V_g$, V_gC_φ) is a bounded operator in the logarithmic Bloch space $\mathcal{LB}$.*

Proof: Suppose $C_\varphi U_g$ is bounded in the logarithmic little Bloch space $\mathcal{LB}_0$. It is clear that for any $f\in\mathcal{LB}$, we have $f_t\in\mathcal{LB}_0$ for any $0<t<1$. Now applying Lemma 5, we get

$$\|C_\varphi U_g(f_t)\|_{\mathcal{L}} \le \|C_\varphi U_g\|\|f_t\|_{\mathcal{L}} \le 4\|C_\varphi U_g\|\|f\|_{\mathcal{L}}.$$

Letting $t\to1^-$, we obtain that

$$\|C_\varphi U_g(f)\|_{\mathcal{L}} \le 4\|C_\varphi U_g\|\|f\|_{\mathcal{L}} < +\infty,$$

which shows $C_\varphi U_g$ is bounded in the logarithmic Bloch space $\mathcal{LB}$. One may similarly prove the boundedness for U_gC_φ, $C_\varphi V_g$, or V_gC_φ. We omit the details here.

3 Boundedness and compactness of $C_\varphi U_g$ on $\mathcal{LB}$ and $\mathcal{LB}_0$

In this section we study the boundedness and compactness of the operator

$$C_\varphi U_g(\text{ or } U_gC_\varphi) : \mathcal{LB}(\text{or } \mathcal{LB}_0) \longrightarrow \mathcal{LB}(\text{or } \mathcal{LB}_0).$$

Theorem 7 *Let φ be an analytic self-map of the unit disc and $g\in H(D)$. Then the following statements hold.*

(i) $C_\varphi U_g : \mathcal{LB}\longrightarrow\mathcal{LB}$ *is bounded if and only if*

$$\sup_{z\in D}(1-|z|^2)\ln\frac{2}{1-|z|}\ln(\ln\frac{2}{1-|\varphi(z)|}) \times|g'(\varphi(z))||\varphi'(z)|<\infty. \quad (1)$$

(ii) $C_\varphi U_g : \mathcal{LB}_0\longrightarrow\mathcal{LB}_0$ *is bounded if and only if (1) holds and*

$$\lim_{|z|\to1^-}(1-|z|^2)\ln(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)|=0. \quad (2)$$

Proof: (i) Assume that $C_\varphi U_g : \mathcal{LB} \longrightarrow \mathcal{LB}$ is bounded. Fix $w \in D$, let

$$f_w(z) = \ln\ln \frac{4}{1-\overline{\varphi(w)}z}. \tag{3}$$

From Lemma 3 we know that $f_w \in \mathcal{LB}$ and $\|f_w\|_{\mathcal{L}} \leq 5$. Since $f_w(\varphi(w)) = \ln\ln \frac{4}{1-|\varphi(w)|^2}$, it follows that

$$\begin{aligned}&(1-|w|^2)\ln(\tfrac{2}{1-|w|})\ln(\ln\tfrac{4}{1-|\varphi(w)|^2})|g'(\varphi(w))||\varphi'(w)|\\ &= (1-|w|^2)\ln(\tfrac{2}{1-|w|})|(C_\varphi U_g f_w)'(w)|\\ &\leq \|C_\varphi U_g\|\|f_w\|_{\mathcal{L}} \leq 5\|C_\varphi U_g\| < +\infty.\end{aligned}$$

Thus (1) holds.

Conversely, suppose that (1) holds. Then, from Lemma 1, we have

$$\begin{aligned}&\|C_\varphi U_g f\|_{\mathcal{LB}}\\ &= \sup_{z\in D}(1-|z|^2)\ln(\tfrac{2}{1-|z|})|f(\varphi(z))g'(\varphi(z))\varphi'(z)|\\ &\leq \sup_{z\in D}(1-|z|^2)\ln(\tfrac{2}{1-|z|})|g'(\varphi(z))\varphi'(z)|\\ &\quad\times(2+\ln\ln\tfrac{2}{1-|\varphi(z)|})\|f\|_{\mathcal{L}}\\ &\leq C\|f\|_{\mathcal{L}}\end{aligned}$$

and

$$\begin{aligned}&|(C_\varphi U_g f)(0)| = |\int_0^{\varphi(0)} f(\zeta)g'(\zeta)\,d\zeta|\\ &\leq \max_{|\zeta|\leq|\varphi(0)|}|f(\zeta)|\max_{|\zeta|\leq|\varphi(0)|}|g'(\zeta)|\\ &\leq (2+\ln\ln\frac{2}{1-|\varphi(0)|})\max_{|\zeta|\leq|\varphi(0)|}|g'(\zeta)|\|f\|_{\mathcal{L}}.\end{aligned}$$

This shows that $C_\varphi U_g$ is bounded.

(ii) Assume $C_\varphi U_g : \mathcal{LB}_0 \longrightarrow \mathcal{LB}_0$ is bounded. Then $C_\varphi U_g : \mathcal{LB} \longrightarrow \mathcal{LB}$ is bounded by Lemma 6, which implies that (1) holds by (i).

Next, We take the test function $f = 1$. It is easily seen that (2) holds.

On the other hand, given any $f \in \mathcal{LB}_0$. If $|\varphi(z)| \longrightarrow 1^-$ as $|z| \to 1^-$, it follows from Lemma 2 and (1) that

$$\begin{aligned}&(1-|z|^2)\ln(\tfrac{2}{1-|z|})|(C_\varphi U_g f)'(z)|\\ &= (1-|z|^2)\ln(\tfrac{2}{1-|z|})|f(\varphi(z))g'(\varphi(z))\varphi'(z)|\\ &\leq C\frac{|f(\varphi(z))|}{\ln\ln\frac{2}{1-|\varphi(z)|}} \longrightarrow 0\end{aligned}$$

as $|z| \to 1^-$.

If $|\varphi(z)| \leq r_0 < 1$ for every $z \in D$, then

$$\begin{aligned}&(1-|z|^2)\ln(\tfrac{2}{1-|z|})|(C_\varphi U_g f)'(z)|\\ &\leq \max_{|w|\leq r_0}|f(w)|(1-|z|^2)\ln(\frac{2}{1-|z|})\\ &\times|g'(\varphi(z))||\varphi'(z)| \longrightarrow 0 \ \ (|z|\to 1^-)\end{aligned}$$

by (2). Hence $C_\varphi U_g f \in \mathcal{LB}_0$ for all $f \in \mathcal{LB}_0$. On the other hand, $C_\varphi U_g$ is bounded in $\mathcal{LB}$ by (i). Hence $C_\varphi U_g$ is a bounded operator in $\mathcal{LB}_0$.

Lemma 8 *Let $C_\varphi U_g$(or $U_g C_\varphi$, $C_\varphi V_g$, $V_g C_\varphi$) : $\mathcal{LB}_0 \longrightarrow \mathcal{LB}_0$ be a bounded operator in $\mathcal{LB}$. Then $C_\varphi U_g$ (or $U_g C_\varphi$, $C_\varphi V_g$, $V_g C_\varphi$) : $\mathcal{LB}_0 \longrightarrow \mathcal{LB}_0$ is compact if and only if for any bounded sequence $\{f_n\}$ in $\mathcal{LB}$ which converges to 0 uniformly on compact subsets of D, we have $\|C_\varphi U_g(f_n)\|_{\mathcal{L}} \longrightarrow 0$ (or $\|U_g C_\varphi(f_n)\|_{\mathcal{L}}$, $\|C_\varphi V_g(f_n)\|_{\mathcal{L}}$, $\|V_g C_\varphi(f_n)\|_{\mathcal{L}} \longrightarrow 0$) as $n \longrightarrow \infty$.*

The result can be proved by Montel theorem, Lemma 1 and 5. The details are omitted here.

Lemma 9 *Let $U \subset \mathcal{LB}_0$. Then U is compact if and only if it is closed, bounded and satisfies*

$$\lim_{|z|\longrightarrow 1}\sup_{f\in U}(1-|z|^2)\ln(\frac{2}{1-|z|})|f'(z)| = 0.$$

The proof is similar to that of Lemma 1 in [5]. The details are omitted.

Theorem 10 *Let φ be an analytic self-map of the unit disc and $g \in H(D)$. Then the following statements hold.*

(i) $C_\varphi U_g : \mathcal{LB} \longrightarrow \mathcal{LB}$ is compact if and only if

$$\begin{aligned}\lim_{|\varphi(z)|\to 1^-}&(1-|z|^2)\ln(\tfrac{2}{1-|z|})\ln(\ln\tfrac{2}{1-|\varphi(z)|})\\ &\times|g'(\varphi(z))||\varphi'(z)| = 0\end{aligned} \tag{4}$$

and

$$\sup_{z\in D}(1-|z|^2)\ln(\frac{2}{1-|z|})|g'(\varphi(z))||\varphi'(z)| < +\infty. \tag{5}$$

(ii) $C_\varphi U_g : \mathcal{LB}_0 \longrightarrow \mathcal{LB}_0$ is compact if and only if

$$\begin{aligned}\lim_{|z|\to 1^-}&(1-|z|^2)\ln(\tfrac{2}{1-|z|})\ln(\ln\tfrac{2}{1-|\varphi(z)|})\\ &\times|g'(\varphi(z))||\varphi'(z)| = 0.\end{aligned} \tag{6}$$

Proof: (i) Assume (4) and (5) hold, which implies (1) holds. Then $C_\varphi U_g : \mathcal{LB} \longrightarrow \mathcal{LB}$ is bounded by Theorem 7. Let $\{f_n\}$ be a bounded sequence in $\mathcal{LB}$

which converges to 0 uniformly on compact subsets of D. We need only to prove $\lim_{n\longrightarrow\infty} \|C_\varphi U_g(f_n)\|_{\mathcal{L}} = 0$ by Lemma 8. This amounts to showing that both

$$\sup_{w\in D}(1-|w|^2)\ln(\tfrac{2}{1-|w|}) \times|f_n(\varphi(w))g'(\varphi(w))\varphi'(w)| \longrightarrow 0$$

and

$$|C_\varphi U_g f_n(0)| \longrightarrow 0.$$

If $|\varphi(w)| > r$, we may assume $r > r_*$, then

$$(1-|w|^2)\ln(\frac{2}{1-|w|})|f_n(\varphi(w))g'(\varphi(w))\varphi'(w)| \le 2\|f_n\|_L(1-|w|^2)\ln(\frac{2}{1-|w|})\ln(\ln\frac{2}{1-|\varphi(w)|}) \times|g'(\varphi(w))||\varphi'(w)|.$$

If $|\varphi(w)| \le r < 1$, by (5), we have

$$(1-|w|^2)\ln(\frac{2}{1-|w|})|f_n(\varphi(w))g'(\varphi(w))\varphi'(w)| \le C\max_{|z|\le r}|f_n(z)|.$$

Thus

$$\sup_{w\in D}(1-|w|^2)\ln(\tfrac{2}{1-|w|})|f_n(\varphi(w))g'(\varphi(w))\varphi'(w)| \le C\max a_{|w|\le r}|f_n(w)| + C\sup_{|\varphi(w)|>r}(1-|w|^2) \times\ln(\tfrac{2}{1-|w|})\ln(\ln\tfrac{2}{1-|\varphi(w)|})|g'(\varphi(w))||\varphi'(w)|.$$

First letting n tend to infinity and subsequently r increase to 1, one obtains that

$$\sup_{w\in D}(1-|w|^2)\ln(\tfrac{2}{1-|w|})|f_n(\varphi(w)) \times g'(\varphi(w))\varphi'(w)| \longrightarrow 0$$

as $n \longrightarrow \infty$.

On the other hand, it is obvious that

$$|C_\varphi U_g f_n(0)| \le \max_{|z|\le|\varphi(0)|}|g'(z)|\max_{|z|\le|\varphi(0)|}|f_n(z)| \le C\max_{|z|\le|\varphi(0)|}|f_n(z)| \longrightarrow 0$$

as $n \longrightarrow \infty$.

Conversely, suppose that $C_\varphi U_g$ is compact in $\mathcal{LB}$. It is obvious that $C_\varphi U_g$ is bounded. Then (1) holds by Theorem 7, which implies that (5) holds. Next, let $\{z_n\}$ be a sequence in D such that $|\varphi(z_n)| \to 1$ as $n \to \infty$. Choose test functions

$$f_n(z) = \frac{1}{a_n}(\ln\ln\frac{4}{1-\overline{\varphi(z_n)}z})^2,$$

where $a_n = \ln\ln\dfrac{4}{1-|\varphi(z_n)|^2}$. It is clear that $f_n(z) \to 0$ uniformly on compact subsets of D. From Lemma 3 and 4, we get $f_n \in \mathcal{LB}$ and $\sup_n \|f_n\|_{\mathcal{L}} < \infty$. Then $\{f_n\}$ is a bounded sequence in $\mathcal{LB}$ which converges to 0 uniformly on compact subsets of D. Noticing that $f_n(\varphi(z_n)) = a_n$, we have

$$\|C_\varphi U_g f_n\|_{\mathcal{L}} \ge \|C_\varphi U_g f_n\|_{\mathcal{LB}} \ge (1-|z_n|^2)\ln(\tfrac{2}{1-|z_n|})|f_n(\varphi(z_n))g'(\varphi(z_n))\varphi'(z_n)| = (1-|z_n|^2)\ln(\tfrac{2}{1-|z_n|}) \times\ln(\ln\tfrac{4}{1-|\varphi(z_n)|^2})|g'(\varphi(z_n))\varphi'(z_n)|.$$

Then

$$\lim_{|\varphi(z)|\to1^-}(1-|z|^2)\ln(\tfrac{2}{1-|z|})\ln(\ln\tfrac{2}{1-|\varphi(z)|}) \times|g'(\varphi(z))||\varphi'(z)| = 0$$

by Lemma 8. Hence (4) holds.

(ii) Assume that (6) holds. Then it implies that (1) and (2) hold, which shows that $C_\varphi U_g : \mathcal{LB}_0 \longrightarrow \mathcal{LB}_0$ is bounded.

Suppose that $f \in \mathcal{LB}_0$ with $\|f\|_{\mathcal{L}} \le 1$. It follows from Lemma 1 that

$$(1-|z|^2)\ln(\tfrac{2}{1-|z|})|(C_\varphi U_g f)'(z)| = (1-|z|^2)\ln(\frac{2}{1-|z|})|g'(\varphi(z))f(\varphi(z))\varphi'(z)| \le (1-|z|^2)\ln(\frac{2}{1-|z|}) \times(2+\ln\ln\frac{2}{1-|\varphi(z)|})|g'(\varphi(z))\varphi'(z)|.$$

Thus

$$\sup\{|(1-|z|^2)\ln(\tfrac{2}{1-|z|})(C_\varphi U_g f)'(z)| : f \in \mathcal{LB}_0, \|f\|_{\mathcal{L}} \le 1\} \le (1-|z|^2)\ln(\frac{2}{1-|z|})(2+\ln\ln\frac{2}{1-|\varphi(z)|}) \times|g'(\varphi(z))\varphi'(z)|,$$

and hence

$$\lim_{|z|\to1^-}\{|(1-|z|^2)\ln(\frac{2}{1-|z|})(C_\varphi U_g f)'(z)| : f \in \mathcal{LB}_0, \|f\|_{\mathcal{L}} \le 1\} = 0,$$

so that $C_\varphi U_g$ is compact in $\mathcal{LB}_0$ by Lemma 9.

Conversely, suppose that $C_\varphi U_g$ is compact in $\mathcal{LB}_0$. From Lemma 9 we have

$$\lim_{|z|\to1^-}\{|(1-|z|^2)\ln(\frac{2}{1-|z|})(C_\varphi U_g f)'(z)| : f \in \mathcal{LB}_0, \|f\|_{\mathcal{L}} \le M\} = 0,$$

for some $M > 0$. Note that the proof of Theorem 7 and the fact that the function given in (3) are in $\mathcal{LB}_0$

and have norms bounded independently of w. We obtain that

$$\lim_{|z|\to 1^-}(1-|z|^2)\ln(\frac{2}{1-|z|})\ln(\ln\frac{2}{1-|\varphi(z)|}) \times|g'(\varphi(z))||\varphi'(z)|=0.$$

The proof of the theorem is completed.

Using the same methods as in the proof of the previous theorems, we can prove the following results.

Theorem 11 *Let φ be an analytic self-map of the unit disc and $g\in H(D)$. Then the following statements hold.*

(i) $U_gC_\varphi:\mathcal{LB}\longrightarrow\mathcal{LB}$ is bounded if and only if

$$\sup_{z\in D}(1-|z|^2)\ln\frac{2}{1-|z|}\ln(\ln\frac{2}{1-|\varphi(z)|})|g'(z)|<+\infty. \tag{7}$$

(ii) $U_gC_\varphi:\mathcal{LB}_0\longrightarrow\mathcal{LB}_0$ is bounded if and only if (7) holds and $g\in\mathcal{LB}_0$.

Theorem 12 *Let φ be an analytic self-map of the unit disc and $g\in H(D)$. Then the following statements hold.*

(i) $U_gC_\varphi:\mathcal{LB}\longrightarrow\mathcal{LB}$ is compact if and only if $g\in\mathcal{LB}$ and

$$\lim_{|\varphi(z)|\to 1^-}[(1-|z|^2)\ln(\tfrac{2}{1-|z|}) \ln(\ln\tfrac{2}{1-|\varphi(z)|})|g'(z)|]=0. \tag{8}$$

(ii) $U_gC_\varphi:\mathcal{LB}_0\longrightarrow\mathcal{LB}_0$ is compact if and only if

$$\lim_{|z|\to 1^-}(1-|z|^2)\ln(\tfrac{2}{1-|z|})\ln(\ln\tfrac{2}{1-|\varphi(z)|}) \times|g'(z)|=0.$$

Taking $\varphi(z)=z$ from Theorem 7, 10, 11, 12, we obtain the following results about the characterization of the boundedness and compactness of the Volterra-type operator $U_g:\mathcal{LB}(\text{ or }\mathcal{LB}_0)\longrightarrow\mathcal{LB}(\text{ or }\mathcal{LB}_0)$.

Corollary 13 *Let $g\in H(D)$. Then*

(i) $U_g:\mathcal{LB}\longrightarrow\mathcal{LB}$ is a bounded operator if and only if $U_g:\mathcal{LB}_0\longrightarrow\mathcal{LB}_0$ is a bounded operator if and only if

$$\sup_{z\in D}(1-|z|^2)\ln\frac{2}{1-|z|}\ln(\ln\frac{2}{1-|z|})|g'(z)|<\infty.$$

(ii) $U_g:\mathcal{LB}\longrightarrow\mathcal{LB}$ is a compact operator if and only if $U_g:\mathcal{LB}_0\longrightarrow\mathcal{LB}_0$ is a compact operator if and only if

$$\lim_{|z|\to 1^-}(1-|z|^2)\ln\frac{2}{1-|z|}\ln(\ln\frac{2}{1-|z|})|g'(z)|=0.$$

4 Boundedness and compactness of $C_\varphi V_g$ on $\mathcal{LB}$ and $\mathcal{LB}_0$

In this section, we characterize the boundedness and compactness of the operator $C_\varphi V_g$(*or* V_gC_φ) : $\mathcal{LB}$(or $\mathcal{LB}_0$) $\longrightarrow\mathcal{LB}$(or $\mathcal{LB}_0$).

Theorem 14 *Let φ be an analytic self-map of the unit disc and $g\in H(D)$. Then the following statements hold.*

(i) $C_\varphi V_g:\mathcal{LB}\longrightarrow\mathcal{LB}$ is bounded if and only if

$$\sup_{z\in D}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|g(\varphi(z))\varphi'(z)|<+\infty. \tag{9}$$

(ii) $C_\varphi V_g:\mathcal{LB}_0\longrightarrow\mathcal{LB}_0$ is bounded if and only if (9) holds and

$$\lim_{|z|\to 1^-}(1-|z|^2)\ln(\frac{2}{1-|z|})|g(\varphi(z))\varphi'(z)|=0. \tag{10}$$

Proof: Suppose $C_\varphi V_g$ is bounded on the logarithmic Bloch space $\mathcal{LB}$. Taking the test function $f(z)=z$, we can easily obtain that

$$\sup_{z\in D}(1-|z|^2)\ln(\frac{2}{1-|z|})|g(\varphi(z))\varphi'(z)|<+\infty. \tag{11}$$

For $\forall\, 0\neq w\in D$, let

$$f_w(z)=\int_0^z(1-\frac{\overline{w}^2}{|w|^2}z^2)^{-1}(\ln\frac{4}{1-\frac{\overline{w}^2}{|w|^2}z^2})^{-1}\,dz. \tag{12}$$

From Lemma 3, we have

$$\sup_{z_1\in D}\frac{(1-|z_1|^2)(\ln\frac{2}{1-|z_1|^2})}{|1-z_1^2||\ln\frac{4}{1-z_1^2}|} <2<+\infty.$$

Applying $z_1=\frac{\overline{w}}{|w|}z$, we obtain that

$$\sup_{z\in D}(1-|z|^2)(\ln\frac{2}{1-|z|^2})|1-\frac{\overline{w}^2}{|w|^2}z^2| \times|\ln\frac{4}{1-\frac{\overline{w}^2}{|w|^2}z^2}|^{-1}<2<+\infty.$$

Hence $f_w\in\mathcal{LB}$ and $\|f_w\|_{\mathcal{L}}<4$ with $w\neq 0$. Then for $w\neq 0$ we obtain that

$$\|C_\varphi V_g(f_w)\|_{\mathcal{LB}}\le\|C_\varphi V_g(f_w)\|_{\mathcal{L}}\le\|C_\varphi V_g\|\|f_w\|_{\mathcal{L}} <4\|C_\varphi V_g\|<+\infty. \tag{13}$$

So for $\forall z \in D$ with $\varphi(z) \neq 0$, applying $w = \varphi(z)$ to (13), we obtain that

$$\sup_{z\in D} \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{4}{1-|\varphi(z)|^2}}|g(\varphi(z))\varphi'(z)|$$
$$= \sup_{z\in D}(1-|z|^2)\ln\frac{2}{1-|z|}|f_w'(\varphi(z))g(\varphi(z))\varphi'(z)|$$
$$= \|C_\varphi V_g(f_w)\|_{\mathcal{LB}} < 4\|C_\varphi V_g\| < \infty$$

For $\forall z \in D$ with $\varphi(z) = 0$, from (11), we have

$$\sup_{z\in D} \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|u(z)\varphi'(z)|$$
$$= \sup_{z\in D}\frac{1}{\ln 2}(1-|z|^2)\ln\frac{2}{1-|z|}|g(\varphi(z))\varphi'(z)|$$
$$< +\infty.$$

Hence (9) holds.

Conversely, suppose that (9) holds. For $f \in \mathcal{LB}$, from Lemma 1, we have

$$\|C_\varphi V_g f\|_{\mathcal{LB}}$$
$$= \sup_{z\in D}(1-|z|^2)\ln(\frac{2}{1-|z|})|f'(\varphi(z))||g(\varphi(z))\varphi'(z)|$$
$$\leq \|f\|_{\mathcal{LB}}\sup_{z\in D}\frac{(1-|z|^2)\ln(\frac{2}{1-|z|})}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|g(\varphi(z))\varphi'(z)|$$
$$\leq C\|f\|_{\mathcal{L}}$$

and

$$|(C_\varphi V_g f)(0)| = |\int_0^{\varphi(0)} f'(\zeta)g(\zeta)\,d\zeta|$$
$$\leq \max_{|\zeta|\leq|\varphi(0)|}|f'(\zeta)|\max_{|\zeta|\leq|\varphi(0)|}|g(\zeta)|$$
$$\leq \frac{\max_{|\zeta|\leq|\varphi(0)|}|g(\zeta)|}{(1-|\varphi(0)|^2)\ln 2}\|f\|_{\mathcal{L}}.$$

This shows that $C_\varphi V_g$ is bounded.

(ii) Assume $C_\varphi V_g : \mathcal{LB}_0 \longrightarrow \mathcal{LB}_0$ is bounded. Then $C_\varphi V_g : \mathcal{LB} \longrightarrow \mathcal{LB}$ is bounded by Lemma 6, which implies that (9) holds by (i).

Next, We take the test function $f = z$. It is easily seen that (10) holds.

Conversely, given $f \in \mathcal{LB}_0$. If $|\varphi(z)| \longrightarrow 1^-$ as $|z| \to 1^-$, it follows from (9) that

$$(1-|z|^2)\ln(\frac{2}{1-|z|})|(C_\varphi V_g f)'(z)|$$
$$= (1-|z|^2)\ln(\frac{2}{1-|z|})|f'(\varphi(z))g(\varphi(z))\varphi'(z)|$$
$$\leq C(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}|f'(\varphi(z))|$$
$$\longrightarrow 0$$

as $|z| \to 1^-$.

If $|\varphi(z)| \leq r_0 < 1$ for every $z \in D$, then, from (10),

$$(1-|z|^2)\ln(\frac{2}{1-|z|})|(C_\varphi U_g f)'(z)|$$
$$\leq \max_{|w|\leq r_0}|f'(w)|(1-|z|^2)\ln(\frac{2}{1-|z|})|g(\varphi(z))\varphi'(z)|$$
$$\longrightarrow 0$$

as $|z| \to 1^-$. Hence $C_\varphi V_g f \in \mathcal{LB}_0$ for any $f \in \mathcal{LB}_0$. Since $C_\varphi V_g$ is bounded on $\mathcal{LB}$ by (i), $C_\varphi U_g$ is bounded on $\mathcal{LB}_0$.

Theorem 15 *Let φ be an analytic self-map of the unit disc and $g \in H(D)$. Then the following statements hold.*

(i) $C_\varphi V_g : \mathcal{LB} \longrightarrow \mathcal{LB}$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1^-}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|g(\varphi(z))\varphi'(z)| = 0. \tag{14}$$

and

$$\sup_{z\in D}(1-|z|^2)\ln(\frac{2}{1-|z|})|g(\varphi(z))\varphi'(z)| < +\infty. \tag{15}$$

(ii) $C_\varphi V_g : \mathcal{LB}_0 \longrightarrow \mathcal{LB}_0$ is compact if and only if

$$\lim_{|z|\to 1^-}\frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|g(\varphi(z))\varphi'(z)| = 0. \tag{16}$$

Proof: (i) Assume (14) and (15) hold, which implies (9) holds. Then $C_\varphi V_g : \mathcal{LB} \longrightarrow \mathcal{LB}$ is bounded by Theorem 14. Let $\{f_n\}$ be a bounded sequence in $\mathcal{LB}$ which converges to 0 uniformly on compact subsets of D. It is clear that the sequence $\{f_n'\}$ converges to 0 uniformly on compact subsets of D. We need only to prove $\lim_{n\to\infty}\|C_\varphi V_g(f_n)\|_{\mathcal{L}} = 0$ by Lemma 8. This amounts to showing that both

$$\sup_{w\in D}(1-|w|^2)\ln(\frac{2}{1-|w|})|f_n'(\varphi(w))$$
$$\times g(\varphi(w))\varphi'(w)| \longrightarrow 0$$

and

$$|C_\varphi V_g f_n(0)| \longrightarrow 0.$$

If $|\varphi(w)| > r$, then

$$(1-|w|^2)\ln(\frac{2}{1-|w|})|f_n'(\varphi(w))g(\varphi(w))\varphi'(w)|$$
$$\leq \|f_n\|_{\mathcal{LB}}\frac{(1-|w|^2)\ln\frac{2}{1-|w|}}{(1-|\varphi(w)|^2)\ln\frac{2}{1-|\varphi(w)|}}$$
$$\times|g(\varphi(w))\varphi'(w)|.$$

If $|\varphi(w)| \le r < 1$, from (15), we have

$$(1-|w|^2)\ln(\frac{2}{1-|w|})|f_n'(\varphi(w))g(\varphi(w))\varphi'(w)| \le C\max_{|z|\le r}|f_n'(z)|.$$

Thus

$$\sup_{w\in D}(1-|w|^2)\ln(\frac{2}{1-|w|})|f_n(\varphi(w))g'(\varphi(w))\varphi'(w)| \le C\max_{|w|\le r}|f_n'(w)| + C\times \max_{|\varphi(w)|>r}\frac{(1-|w|^2)\ln\frac{2}{1-|w|}}{(1-|\varphi(w)|^2)\ln\frac{2}{1-|\varphi(w)|}}|g(\varphi(w))\varphi'(w)|.$$

First letting n tend to infinity and subsequently r increase to 1, one obtains that

$$\sup_{w\in D}(1-|w|^2)\ln(\frac{2}{1-|w|})|f_n'(\varphi(w)) \times g(\varphi(w))\varphi'(w)| \longrightarrow 0$$

as $n \longrightarrow \infty$.

On the other hand, it is obvious that

$$\begin{aligned}|C_\varphi V_g f_n(0)| &\le \max_{|z|\le|\varphi(0)|}|g(z)|\max_{|z|\le|\varphi(0)|}|f_n'(z)| \\ &\le C\max_{|z|\le|\varphi(0)|}|f_n'(z)| \longrightarrow 0\end{aligned}$$

as $n \longrightarrow \infty$.

Conversely, suppose that $C_\varphi V_g$ is compact on $\mathcal{LB}$. It is obvious that $C_\varphi V_g$ is bounded. Then (9) holds by Theorem 7, which implies that (15) holds.

Next assume that (14) fails. Then there exists a subsequence $\{z_n\} \subset D$ and an $\epsilon_0 > 0$ such that $|\varphi(z_n)| \to 1 (n \to \infty)$ and

$$\frac{(1-|z_n|^2)\ln\frac{2}{1-|z_n|}}{(1-|\varphi(z_n)|^2)\ln\frac{2}{1-|\varphi(z_n)|}}|\varphi'(z_n)g(\varphi(z_n))| \ge \epsilon_0.$$

Let $\varphi(z_n) = r_n e^{i\theta_n}$. We take

$$f_n(z) = \int_0^z (\frac{r_n}{1-e^{-i\theta_n}r_n w} - \frac{r_n^2}{1-r_n^2 e^{-i\theta_n}w}) \times(\ln\frac{4}{1-r_n^2e^{-i\theta_n}w})^{-1}\,dw.$$

Then

$$f_n'(z) = (\frac{r_n}{1-e^{-i\theta_n}r_n z} - \frac{r_n^2}{1-r_n^2e^{-i\theta_n}z}) \times(\ln\frac{4}{1-r_n^2e^{-i\theta_n}z})^{-1}.$$

One can obtain that

$$|f_n(z)| \le \frac{1-r_n}{(1-|z|)^2}(\ln\frac{4}{1+|z|})^{-1}$$

by a direct calculation and $\|f_n\|_{\mathcal{L}} \le 8$ by Lemma 3 and 4. Then $\{f_n\}$ is a bounded sequence in $\mathcal{LB}$ which converges to 0 uniformly on compact subsets of D.

On the other hand, for enough large n, we have

$$\begin{aligned}&\|C_\varphi V_g(f_n)\|_{\mathcal{L}} \\ &\ge (1-|z_n|^2)\ln\frac{2}{1-|z_n|}|f_n'(\varphi(z_n))| \\ &\quad\times|\varphi'(z_n)g((\varphi(z_n))| \\ &= (1-|z_n|^2)\ln\frac{2}{1-|z_n|} \\ &\quad\times(\frac{r_n}{1-r_n^2} - \frac{r_n^2}{1-r_n^3})(\ln\frac{4}{1-r_n^3})^{-1}\times \\ &\quad|\varphi'(z_n)g((\varphi(z_n))| \\ &\ge \frac{(1-|z_n|^2)\ln\frac{2}{1-|z_n|}}{6(1-|\varphi(z_n)|^2)\ln\frac{2}{1-|\varphi(z_n)|}} \\ &\quad\times|\varphi'(z_n)g((\varphi(z_n))| \\ &\ge \frac{\epsilon_0}{6} \quad (n\to\infty).\end{aligned}$$

This contradicts the compactness of $C_\varphi V_g$ by Lemma 8. Hence (14) holds.

(ii) Assume that (16) holds. Then it implies that (9) and (10) hold, which shows that $C_\varphi U_g : \mathcal{LB}_0 \to \mathcal{LB}_0$ is bounded.

Suppose that $f \in \mathcal{LB}_0$ with $\|f\|_{\mathcal{L}} \le 1$. Then we have

$$\begin{aligned}&(1-|z|^2)\ln(\frac{2}{1-|z|})|(C_\varphi V_g f)'(z)| \\ &= (1-|z|^2)\ln(\frac{2}{1-|z|})|f'(\varphi(z))g(\varphi(z))\varphi'(z)| \\ &\le \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|g(\varphi(z))\varphi'(z)|.\end{aligned}$$

Thus

$$\begin{aligned}&\sup\{|(1-|z|^2)\ln(\frac{2}{1-|z|})(C_\varphi V_g f)'(z)| \\ &: f\in\mathcal{LB}_0, \|f\|_{\mathcal{L}} \le 1\} \\ &\le \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|g(\varphi(z))\varphi'(z)|,\end{aligned}$$

and

$$\lim_{|z|\to1^-}\sup\{(1-|z|^2)\ln(\frac{2}{1-|z|})|(C_\varphi V_g f)'(z)| : f\in\mathcal{LB}_0, \|f\|_{\mathcal{L}} \le 1\} = 0.$$

This implies that $C_\varphi V_g$ is compact in $\mathcal{LB}_0$ by Lemma 9.

Conversely, suppose that $C_\varphi V_g$ is compact in $\mathcal{LB}_0$. From Lemma 9 we have

$$\lim_{|z|\to1^-}\sup\{(1-|z|^2)\ln(\frac{2}{1-|z|})|(C_\varphi V_g f)'(z)| : f\in\mathcal{LB}_0, \|f\|_{\mathcal{L}} \le M\} = 0,$$

for some $M > 0$. Note that the proof of Theorem 14 and the fact that the function given in (12) are in $\mathcal{LB}_0$ and have norms bounded independently of w. We obtain that

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|g(\varphi(z))\varphi'(z)| = 0$$

for $\varphi(z) \neq 0$. However, if $\varphi(z) = 0$, it follows from (10) that

$$\lim_{|z|\to 1^-}(1-|z|^2)\ln\frac{2}{1-|z|}|g(\varphi(z))\varphi'(z)| = 0.$$

The proof of the theorem is completed.

Using the same methods as in the proof of Theorem 14 and 15, we can prove the following results.

Theorem 16 *Let φ be an analytic self-map of the unit disc and $g \in H(D)$. Then the following statements hold.*

(i) $V_gC_\varphi : \mathcal{LB} \longrightarrow \mathcal{LB}$ is bounded if and only if

$$\sup_{z\in D} \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|g(z)\varphi'(z)| < +\infty. \quad (17)$$

(ii) $V_gC_\varphi : \mathcal{LB}_0 \longrightarrow \mathcal{LB}_0$ is bounded if and only if (17) holds and

$$\lim_{|z|\to 1^-}(1-|z|^2)\ln\frac{2}{1-|z|}|g(z)\varphi'(z)| = 0.$$

Theorem 17 *Let φ be an analytic self-map of the unit disc and $g \in H(D)$. Then the following statements hold.*

(i) $V_gC_\varphi : \mathcal{LB} \longrightarrow \mathcal{LB}$ is compact if and only if

$$\lim_{|\varphi(z)|\to 1^-} \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|g(z)\varphi'(z)| = 0.$$

and

$$\sup_{z\in D}(1-|z|^2)\ln(\frac{2}{1-|z|})|g(z)\varphi'(z)| < +\infty.$$

(ii) $V_gC_\varphi : \mathcal{LB}_0 \longrightarrow \mathcal{LB}_0$ is compact if and only if

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|g(z)\varphi'(z)| = 0.$$

Taking $\varphi(z) = z$, from Theorem 14, 15, we obtain the following results about the characterization of the boundedness and compactness of the Volterra-type operator $V_g : \mathcal{LB}(\text{ or } \mathcal{LB}_0) \to \mathcal{LB}(\text{ or } \mathcal{LB}_0)$.

Corollary 18 *Let $g \in H(D)$. Then*

(i) $V_g : \mathcal{LB} \to \mathcal{LB}$ is a bounded operator if and only if $V_g : \mathcal{LB}_0 \to \mathcal{LB}_0$ is a bounded operator if and only if $g \in H^\infty$, where H^∞ denotes the algebra of bounded analytic functions in D.

(ii) $V_g : \mathcal{LB} \to \mathcal{LB}$ is a compact operator if and only if $V_g : \mathcal{LB}_0 \to \mathcal{LB}_0$ is a compact operator if and only if $g \equiv 0$.

Taking $g(z) = 1$, from Theorem 16, 17, we obtain the following results.

Corollary 19 *Let φ be an analytic self-map of D. Then*

(i) C_φ is a bounded operator in $\mathcal{LB}$ if and only if

$$\sup_{z\in D} \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)| < +\infty. \quad (18)$$

(ii) C_φ is a bounded operator in $\mathcal{LB}_0$ if and only if $\varphi \in \mathcal{LB}_0$ and (18) holds.

(iii) C_φ is a compact operator in $\mathcal{LB}$ if and only if

$$\lim_{|\varphi(z)|\to 1^-} \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)| = 0.$$

(iv) C_φ is a compact operator in $\mathcal{LB}_0$ if and only if

$$\lim_{|z|\to 1^-} \frac{(1-|z|^2)\ln\frac{2}{1-|z|}}{(1-|\varphi(z)|^2)\ln\frac{2}{1-|\varphi(z)|}}|\varphi'(z)| = 0.$$

The facts (i) and (iii) here are proved in Theorem 1 and Theorem 2 of [25].

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