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A Novel Analytical Solution of Radial Diffusion Equation with Constant Terminal Rate of Slightly Compressible Fluid

LUSJEN ISMAILI
Ministry of Infrastructure and Energy,
Tirana,
ALBANIA

Abstract: - In reservoir engineering, the flow and type of fluids crossing porous media, are associated with pressure drops leading to variations on hydrodynamic parameters and the filtration stages which are of interest to be assessed as they impact the production efficiency of oil and gas industry. These parameters are the key to the practical solutions of numerous subjects faced during oil field exploitation. Dealing with the problem of sound field development initiatives requires a rigorous examination of unsteady filtration of slightly compressible fluid in the reserves layer. The state of the art of the proposed scientific work is to present an innovative mathematical approach that gives unique results, in the mathematical connection and combination of the equations used versus existing diffusion equation in the case of the constant terminal rate solution. This model helps the designers in the field of oil and gas to better and faster evaluate the diffusivity of the pay zones, the different hydrodynamic parameters, and the different variables that take part in the development of fluid filtration processes in the porous medium expressed as in the dependence of time, distance, and other variables, all of which together impact the well testing and long-term projections.

Key-Words: - compressibility, unsteady-state flow, reservoir, pay zone, hydrodynamic, parameter, filtration, flow rate, porous media, equation.

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1 Introduction

In the oil and gas industry, considering the cases when the pay zone is still unopened, the pressure at any point within the bed can be assumed as constant. Analogously can be accepted for density as a state parameter. With the opening of the pay zone from a well, from the moment of well completion, it will begin to drain and, due to the release of some fluid from the producing formation, the pressure will begin to drop. With the cleavage of the pay zone from a well, starting from the well completion, draining issues, and, due to fluid discharge from the producing formation, the pressure will decrease in time. As fluid withdrawal continues, the decrease in pressure will propagate further from the well in the direction of the reservoir boundary as stated by fluids law. These fluid properties will have a direct impact, especially in the field of oil and gas, and may prohibit the exploitation of wells with high efficiency. Practically, the filtration will be unstable and the radius of influence of the well will constantly increase as stated in the studies, [1], [2]. During this time when the pressure changes at a certain rate, i.e. it does not remain constant but is constantly changing, the flow state is known to be unsteady-state flow. In conditions where the flow is in an unstable state, the flow rate into a representative volume of a porous medium is not equal to the flow rate leaving from this element of volume of porous media. Based on these pressure changes and if the probe radius of investigation has not reached the boundary of the reservoir, i.e., the reservoir will act as if it were of infinite size, it can be said that the flow in the unsteady state is defined as the time during which the boundary does not affect the pressure behavior in the reservoir. The period, during which the process of increasing the radius of influence, approaching the radius of the contour (or of the drainage area), is scientifically known as the first stage of filtering for an unsteady-state flow layer, [3]. Also, in cases when fluid quantity entering from the feeding area to the zone of production is less than what is leaving, the pressure of the layer (in the contour) will begin to decrease, [4].

Consequently, the filtering process is becoming unsteady as theoretically expected. This fraction of time is related to the second stage of filtration for an unsteady-state flow, [1], [2]. Considering the first case, the unsteady filtering stage can be considered as a series of settled states and then move on to solving the problem as mentioned in the respective studies, [3], [4], [5]. On the other hand, when the hydrodynamic study of the well refers to the unsteady regime for a relatively short time of process development; then the implementation of the unsteady filtering replacement method with a series of steady stages, leads to hefty errors, [3], [4]. In such conditions, it is obligatory and indispensable to delve into this concern in more detail by employing new mathematical approaches with more parameters of influence.

Strongly affected by the above-mentioned issues and uncertainties, then the exploitation of a layer from a central well with constant flow in the filtration conditions for unsteady flow regimes is suggested. Afterwards using a new mathematical technique, the terminal constant rate solution of the radial diffusion equation is established. The solution of the diffusion equation with constant terminal rate taking into account the entire flowing time was first presented in 1949 [6] using Laplace transforms for both the constant terminal [7] rate and constant terminal pressure cases, as well as by [8] for a well situated within a no-flow boundary for each flow time value, as well as for all the geometrical configurations. In the solution presented by them, three conditions are considered; the initial state in which the pressure anywhere within the drainage volume is equal to the initial equilibrium pressure p, as well as two boundary conditions which are:

The first is the pressure at the outer, infinite boundary that is not affected by the pressure disturbance at the wellbore and vice versa, and the second is the line source inner boundary condition. They also use the Boltzmann transformation, the diffusivity constant and the substitution of the parameters taken into consideration by them.

The approach presented and the conclusions obtained from our analysis are based on the initial condition given in expression (i), the boundary condition given in expression (ii), the piezometric conductivity, the parameter x which is expressed as a ratio that relates the two variables r and i which is given in Eq 5, the parameter y which expresses the change in pressure depending on the parameter x, as well as the three variables $\frac{\partial P}{\partial t}$; $\frac{\partial P}{\partial r}$; and $\frac{\partial^2 P}{\partial r^2}$:
All these parameters taken into consideration and their mathematical relationship expressed based on the physical concept of fluid mobility in the porous medium make it possible not only to solve the diffusion equation in a different and simple mathematical method, but also the variables that take parts in this equation, which are expressed as a function of different variables, help to solve many problems encountered in the testing and hydrodynamic analysis of wells as given in the studies, [9], [10], [11], [12], [13].

From the literature we know that the diffusivity equation is a combination of three physical principles; the continuity equation, Darcy's law and the equation of state regarding a slightly compressible liquid, [14], [15], [16]. Employing the continuity equation, we can express velocities of the flowing fluid for the case of three-direction system (Eq.2):

$$\nabla(\rho \cdot V) = -\frac{\partial}{\partial t}(\rho \cdot \emptyset); \tag{1}$$

The differential forms of the equation of motion for the case of three dimensional can be given from mathematical expression in Eq.2:

$$\begin{aligned} v_{\times} &= -\frac{k}{\mu} \cdot \frac{\partial P}{\partial \times}; \\ v_{\gamma} &= -\frac{k}{\mu} \cdot \frac{\partial P}{\partial \gamma}; \\ v_{z} &= -\frac{k}{\mu} \cdot \frac{\partial P}{\partial z}; \end{aligned} \tag{2}$$

Likewise, the equation of state for the case of a fluid is given and represented by the mathematical expression in Eq.3:

$$C = -\frac{1}{V} * \frac{\partial V}{\partial P} \tag{3}$$

On the other hand, the formulation of the equation for the filtration of slightly compressible fluids in isotropic porous media is reached and can be represented by the mathematical expression in Eq.4 merging Eq.1, Eq.2 and Eq.3. [2], [4], [17].

$$\nabla^2 \rho = \frac{1}{\aleph} \cdot \frac{\partial \rho}{\partial t} \tag{4}$$

where: $\aleph = \frac{k}{\emptyset \cdot \cdot C}$ → represents piezometric conductivity.

Based on Eq. 4, performing the transformation of coordinates from polar to Cartesian (Laplace and Fourier transforms) [18], [19] combining and replacing different equations, as well as considering two variables given in the following mathematical expression below in Eq.5 and Eq.6 (represent my assumptions) we can evaluate them for first and second derivative as a function of r: $\frac{\partial P}{\partial r}$; $\frac{\partial^2 P}{\partial r^2}$

$$\times = \frac{\mathrm{r}^2}{\aleph \cdot \mathrm{t}} \quad (5)$$

and

$$\gamma = \frac{\partial \mathrm{P}}{\partial \times} \quad (6)$$

Using the above assumptions we have succeeded in solving the diffusion equation for the case of constant terminal rate solution, applying a new, simple, flexible, and mathematical technique never applied in other oil and gas studies.

2 Methodology

Initially, we assume an oil-bearing bed with the same thickness h, having an infinite extent and initial reservoir pressure P_i. This layer is exploited by a well with a constant flow rate "Q" with the focus of examining the pressure distribution in space and time.

In our approach as an initial condition governed by the expression given in *(i)* is employed:

$$P = P_i \; for\; each\; r > r_w \; and\; t = 0 \quad (i)$$

Afterwards the expression given in (ii) serves as a boundary condition:

$$\lim P = P_i$$
$$for\; r \to \infty \; and\; for\; each\; t > 0 \quad (ii)$$

Practically the radial flow conditions should be expressed with cylindrical coordinates correlated to cartesian coordinates as given in Figure 1, [2], [3].

Following the transformation from cylindrical coordinates to cartesian the mathematical expressions given in *(iii)* are carried out:

$$\frac{\times}{r} = \cos\theta \rightarrow \times = r \cdot \cos\theta$$
$$\frac{\gamma}{r} = \sin\theta \rightarrow \times = r \cdot \sin\theta$$
$$\begin{cases} \times^2 = \mathrm{r}^2 \cdot \cos^2\theta \\ \gamma^2 = \mathrm{r}^2 \cdot \sin^2\theta \end{cases} \quad (iii)$$
$$\times^2 + \gamma^2 = r^2 \cdot (cos^2\theta + sin^2\theta)$$
$$\times^2 + \gamma^2 = r^2 \rightarrow r = (\times^2 + \gamma^2)^{\frac{1}{2}}$$

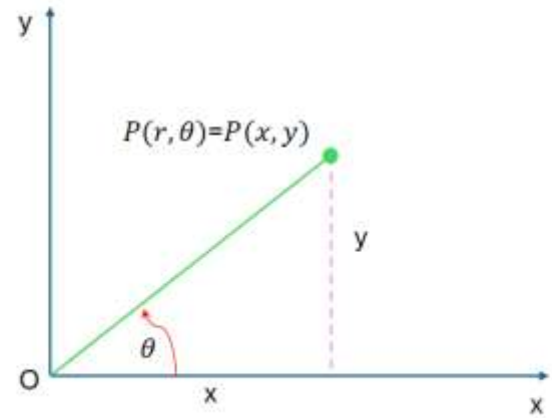

Fig. 1: Cylindrical Coordinate to Cartesian Coordinate. Adapted after [20]

From Eq. 4 and the transformation given in (iii), the solution of the problem can be as given in Eq.7:

$$\nabla^2 \rho = \frac{1}{\aleph} \cdot \frac{\partial \rho}{\partial \mathrm{t}}$$
$$\frac{\partial^2 \rho}{\partial \times^2} + \frac{\partial^2 \rho}{\partial \gamma^2} = \frac{1}{\aleph} \cdot \frac{\partial \rho}{\partial t}$$
$$\frac{\partial^2 \rho}{\partial \times^2} = \frac{\partial}{\partial \times}\left(\frac{\partial \rho}{\partial \times}\right)$$
$$\frac{\partial \rho}{\partial \times} = \frac{\partial \rho}{\partial r} \cdot \frac{\partial r}{\partial \times}$$
$$\frac{\partial r}{\partial \times} = \frac{\partial}{\partial \times}(\times^2 + \gamma^2)^{\frac{1}{2}}$$
$$\frac{\partial r}{\partial \times} = \frac{1}{2}(\times^2 + \gamma^2)^{\frac{1}{2}-1} \cdot 2\times$$
$$\frac{\partial r}{\partial \times} = \frac{1}{\times}(\times^2 + \gamma^2)^{-\frac{1}{2}} \quad (7)$$
$$\frac{\times}{(\times^2 + \gamma^2)^{\frac{1}{2}}} = \frac{\times}{r}$$
$$\frac{\partial \rho}{\partial \times} = \frac{\partial \rho}{\partial r} \cdot \frac{\partial r}{\partial \times} = \frac{\partial \rho}{\partial r} \cdot \frac{\times}{r}$$
$$\frac{\partial^2 \rho}{\partial \times^2} = \frac{\partial}{\partial \times}\left(\frac{\partial \rho}{\partial \times}\right) = \frac{\partial}{\partial \times}\left(\frac{\partial \rho}{\partial r} \cdot \frac{\times}{r}\right)$$
$$\frac{\partial}{\partial \times}\left(\frac{\partial \rho}{\partial r}\right) \cdot \frac{\times}{r} + \frac{\partial}{\partial \times}\left(\frac{\times}{r}\right) \cdot \frac{\partial \rho}{\partial r}$$
$$\frac{\partial^2 \rho}{\partial \times^2} = \frac{\partial}{\partial r}\left(\frac{\partial \rho}{\partial r}\right) \cdot \frac{\partial r}{\partial \times} \cdot \frac{\times}{r} + \left(\frac{r - \frac{\partial r}{\partial \times} \cdot \times}{r^2}\right) \cdot \frac{\partial \rho}{\partial r}$$

$$\frac{\partial^2\rho}{\partial\times^2}=\frac{\partial^2\rho}{\partial r^2}\cdot\frac{\times}{r}\cdot\frac{\times}{r}+\left(\frac{r-\frac{\times}{r}\cdot\times}{r^2}\right)\cdot\frac{\partial\rho}{\partial r}$$
$$\frac{\partial^2\rho}{\partial\times^2}=\frac{\partial^2\rho}{\partial r^2}\cdot\frac{\times^2}{r^2}+\left(\frac{r^2-\times^2}{r^3}\right)\cdot\frac{\partial\rho}{\partial r}$$
$$=\frac{\partial^2\rho}{\partial r^2}\cdot\frac{\times^2}{r^2}+\left(\frac{\times^2+\gamma^2-\times^2}{r^3}\right)\cdot\frac{\partial\rho}{\partial r}$$
$$\frac{\partial^2\rho}{\partial\times^2}=\frac{\partial^2\rho}{\partial r^2}\cdot\frac{\times^2}{r^2}+\frac{\gamma^2}{r^3}\cdot\frac{\partial\rho}{\partial r}$$

Applying more transformation than the following mathematical expression in Eq. 8 can be obtained:

$$\frac{\partial^2\rho}{\partial\times^2}=\frac{\partial^2\rho}{\partial r^2}\cdot\frac{\times^2}{r^2}+\frac{\gamma^2}{r^3}\cdot\frac{\partial\rho}{\partial r}$$
$$\frac{\partial^2\rho}{\partial\gamma^2}=\frac{\partial^2\rho}{\partial r^2}\cdot\frac{\gamma^2}{r^2}+\frac{\times^2}{r^3}\cdot\frac{\partial\rho}{\partial r} \tag{8}$$

Applying and merging the above expression with their influencing parameters from (Eq.8) then the expression Eq.9 is obtained.

$$\frac{\partial^2\rho}{\partial\times^2}+\frac{\partial^2\rho}{\partial\gamma^2}=$$
$$=\frac{\partial^2\rho}{\partial r^2}\cdot\frac{\times^2}{r^2}+\frac{\gamma^2}{r^3}\cdot\frac{\partial\rho}{\partial r}+\frac{\partial^2\rho}{\partial r^2}\cdot\frac{\gamma^2}{r^2}+\frac{\times^2}{r^3}\cdot\frac{\partial\rho}{\partial r}$$
$$=\left(\frac{\times^2}{r^2}+\frac{\gamma^2}{r^2}\right)\cdot\frac{\partial^2\rho}{\partial r^2}+\left(\frac{\times^2}{r^3}+\frac{\gamma^2}{r^3}\right)\cdot\frac{\partial\rho}{\partial r} \tag{9}$$
$$=\left(\frac{r^2}{r^2}\right)\cdot\frac{\partial^2\rho}{\partial r^2}+\left(\frac{r^2}{r^3}\right)\cdot\frac{\partial\rho}{\partial r}$$
$$\therefore\frac{\partial^2\rho}{\partial\times^2}+\frac{\partial^2\rho}{\partial\gamma^2}=\frac{\partial^2\rho}{\partial r^2}+\frac{1}{r}\frac{\partial\rho}{r\cdot\partial r}$$

Based on Eq.9 we simply write as following Eq. 10:

$$\nabla^2\rho=\frac{\partial^2\rho}{\partial\times^2}+\frac{\partial^2\rho}{\partial\gamma^2}$$
$$=\frac{1}{\aleph}\cdot\frac{\partial\rho}{\partial t}\leftrightarrow\frac{\partial^2\rho}{\partial r^2}+\frac{1}{r}\frac{\partial\rho}{*\partial r}=\frac{1}{\aleph}\cdot\frac{\partial\rho}{\partial t} \tag{10}$$

The equation of state for the case of a compressible liquid as represented in Eq. 11 is used: in respect to initial density values.

$$\rho=\rho_0\cdot e^{C\times(P-P_0)}\text{ or}$$
$$\rho=\rho_0\cdot[1+C\cdot(P_0-P)] \tag{11}$$

Applying the first derivative of the density function with respect to r, we get the mathematical expression as given in Eq.12:

$$\frac{\partial\rho}{\partial r}=\frac{\partial}{\partial r}\left(\rho_0+\rho_0\cdot C\cdot(P_0-P)\right) \tag{12}$$
$$\frac{\partial\rho}{\partial r}=\rho_0\cdot C\cdot\frac{\partial P}{\partial r}$$
$$\nabla\rho=\rho_0\cdot C\cdot\nabla P$$
$$\nabla P=\frac{1}{\rho_0\cdot C}\cdot\nabla\rho$$

Applying the second derivative of the above function (Eq.12) to the mathematical expression we get Eq.13.

$$\frac{\partial^2\rho}{\partial r^2}=\rho_0\cdot C\cdot\frac{\partial^2 P}{\partial r^2}$$
$$\frac{\partial\rho}{\partial t}=\rho_0\cdot C\cdot\frac{\partial P}{\partial t} \tag{13}$$

By substituting the expression in Eq.13 in Eq. 10, then the relationship as given in Eq.14 can be achieved.

$$\rho_0\cdot C\cdot\frac{\partial^2 P}{\partial r^2}+\frac{1}{r}\cdot\rho_0\cdot C\cdot\frac{\partial P}{\partial r}=$$
$$=\frac{1}{\aleph}\cdot\rho_0\cdot C\cdot\frac{\partial P}{\partial t}$$
$$=\rho_0\cdot C\cdot\left(\frac{\partial^2 P}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial P}{\partial r}\right) \tag{14}$$
$$=\rho_0\cdot C\cdot\left(\frac{1}{\aleph}\cdot\frac{\partial P}{\partial t}\right)$$
$$\rightarrow\frac{\partial^2 P}{\partial r^2}+\frac{1}{r}\cdot\frac{\partial P}{\partial r}=\frac{1}{\aleph}\cdot\frac{\partial P}{\partial t}$$

3 Results and Discussion

To simplify the analytical solution of the problem raised, we have considered, without spoiling the solution, that the radius of the well is inefficiently small, thus having a suction point. Then we have marked with x the ratio that connects the two variables s and t and continuing with its substitution in equation 14, the result was obtained as following for three variables $\frac{\partial P}{\partial t}$; $\frac{\partial P}{\partial r}$; and $\frac{\partial^2 P}{\partial r^2}$:

$$\times=\frac{r^2}{\aleph\cdot t}$$
$$\frac{\partial P}{\partial t}=\frac{\partial P}{\partial\times}\cdot\frac{\partial\times}{\partial t}$$
$$\frac{\partial P}{\partial t}=\frac{\partial P}{\partial\times}\cdot\frac{\partial}{\partial t}\left(\frac{r^2}{\aleph\cdot t}\right) \tag{15}$$
$$\frac{\partial P}{\partial t}=\frac{\partial P}{\partial\times}\cdot\left[\frac{r^2}{\aleph}\cdot\left(-\frac{1}{t^2}\right)\right]$$
$$\frac{\partial P}{\partial t}=\frac{\partial P}{\partial\times}\cdot\left(-\frac{r^2}{\aleph\cdot t^2}\right)$$

$$\rightarrow \frac{1}{\aleph}\cdot\frac{\partial P}{\partial t} = -\frac{r^2}{\aleph\cdot t^2}\cdot\frac{\partial P}{\partial \times}$$

For $\frac{\partial \boldsymbol{P}}{\partial \boldsymbol{r}}$ the mathematical relationship as following in Eq.16 can be found.

$$\begin{aligned}
&\frac{\partial P}{\partial r} = \frac{\partial P}{\partial \times}\cdot\frac{\partial \times}{\partial r} = \\
&= \frac{\partial P}{\partial \times}\cdot\frac{\partial}{\partial r}\left(\frac{r^2}{\aleph\cdot t}\right)\\
&= \frac{\partial P}{\partial \times}\cdot\left(\frac{2r}{\aleph\cdot t}\right)\\
&\frac{1}{\mathrm{r}}\cdot\frac{\partial \mathrm{P}}{\partial \mathrm{r}} = \frac{1}{r}\cdot\frac{2r}{\aleph\cdot t}\cdot\frac{\partial P}{\partial \times}\\
&\rightarrow \frac{1}{r}\cdot\frac{\partial P}{\partial r} = \frac{2}{\aleph\cdot t}\cdot\frac{\partial P}{\partial \times}
\end{aligned} \tag{16}$$

For $\frac{\partial^2 \boldsymbol{P}}{\partial \boldsymbol{r}^2}$ the mathematical relationship as following in Eq.17 can be carried out:

$$\begin{aligned}
&\frac{\partial^2 P}{\partial r^2} = \frac{\partial}{\partial r}\left(\frac{\partial P}{\partial r}\right) = \frac{\partial}{\partial r}\left(\frac{2r}{\aleph\cdot t}\cdot\frac{\partial P}{\partial \times}\right)\\
&= \frac{\partial}{\partial r}\left(\frac{2r}{\aleph\cdot t}\right)\cdot\frac{\partial P}{\partial \times} + \frac{\partial}{\partial r}\left(\frac{\partial P}{\partial \times}\right)\cdot\frac{2r}{\aleph\cdot t}\\
&= \frac{2}{\aleph\cdot t}\cdot\frac{\partial P}{\partial \times} + \frac{\partial}{\partial \times}\cdot\left(\frac{\partial P}{\partial \times}\right)\cdot\frac{\partial \times}{\partial r}\cdot\frac{2r}{\aleph\cdot t}\\
&= \frac{2}{\aleph\cdot t}\cdot\frac{\partial P}{\partial \times} + \frac{\partial^2 P}{\partial \times^2}\cdot\frac{\partial}{\partial r}\left(\frac{r^2}{\aleph\cdot t}\right)\cdot\frac{2r}{\aleph\cdot t}\\
&= \frac{2}{\aleph\cdot t}\cdot\frac{\partial P}{\partial \times} + \frac{\partial^2 P}{\partial \times^2}\cdot\frac{4r^2}{\aleph^2\cdot t^2}
\end{aligned} \tag{17}$$

By substituting equations 15, 16 and 17 into equation 14, we get the result as following in Eq.18:

$$\begin{aligned}
&\frac{\partial^2 \rho}{\partial \mathrm{r}^2} + \frac{1}{\mathrm{r}}\frac{\partial \rho}{\cdot\partial \mathrm{r}} = \frac{1}{\aleph}\cdot\frac{\partial \rho}{\partial \mathrm{t}}\\
&= \frac{2}{\aleph\cdot t}\cdot\frac{\partial P}{\partial \times} + \frac{\partial^2 P}{\partial \times^2}\cdot\frac{4r^2}{\aleph^2\cdot t^2} + \frac{2}{\aleph\cdot t}\cdot\frac{\partial P}{\partial \times}\\
&= -\frac{r^2}{\aleph^2\cdot t^2}\cdot\frac{\partial P}{\partial \times}\\
&\frac{4r^2}{\aleph^2\cdot t^2}\cdot\frac{\partial^2 P}{\partial \times^2} + \frac{4}{\aleph\cdot t}\cdot\frac{\partial P}{\partial \times} + \frac{r^2}{\aleph^2\cdot t^2}\cdot\frac{\partial P}{\partial \times} = 0\\
&= \frac{4r^2}{\aleph^2\cdot t^2}\cdot\frac{\partial^2 P}{\partial \times^2} + \left(\frac{4}{\aleph\cdot t} + \frac{r^2}{\aleph^2\cdot t^2}\right)\cdot\frac{\partial P}{\partial \times} = 0
\end{aligned} \tag{18}$$

Then both sides are multiplied of the Eq.18 by $\aleph\cdot t$ and considering that $\times = \frac{\mathbf{r^2}}{\aleph\cdot\mathbf{t}}$ than we can simply get the result as given in Eq. 19:

$$\frac{4r^2}{\aleph\cdot t}\cdot\frac{\partial^2 P}{\partial \times^2} + \left(4 + \frac{r^2}{\aleph * t}\right)\cdot\frac{\partial P}{\partial \times} = 0 \tag{19}$$

$$\begin{aligned}
&4\times\cdot\frac{\partial^2 P}{\partial \times^2} + (4+\times)\cdot\frac{\partial P}{\partial \times} = 0\\
&\rightarrow 4\times\cdot\frac{\partial^2 P}{\partial \times^2} = -(4+\times)\cdot\frac{\partial P}{\partial \times}\\
&\rightarrow \frac{\partial^2 P}{\partial \times^2} = -\left(\frac{4+\times}{4\times}\right)\\
&\therefore \frac{\partial^2 \boldsymbol{P}}{\partial \times^2} = \left(-\frac{\mathbf{1}}{\times} - \frac{\mathbf{1}}{\mathbf{4}}\right)\cdot\frac{\partial \boldsymbol{P}}{\partial \times}
\end{aligned}$$

Supposing the assumptions that $\gamma = \frac{\partial P}{\partial \times}$ and extending our calculation it is possible to get the mathematical expression as given in Eq. 20:

$$\begin{aligned}
&\frac{\partial^2 P}{\partial \times^2} = \frac{\partial}{\partial \times}\cdot\left(\frac{\partial P}{\partial \times}\right) = \left(-\frac{1}{\times} - \frac{1}{4}\right)\cdot\frac{\partial P}{\partial \times}\\
&\frac{\partial \gamma}{\partial \times} = \left(-\frac{1}{\times} - \frac{1}{4}\right)\cdot\gamma\\
&\frac{\partial \gamma}{\partial \gamma} = \left(-\frac{1}{\times} - \frac{1}{4}\right)\cdot\partial\times\\
&\frac{d\gamma}{\gamma} = \left(-\frac{1}{\times} - \frac{1}{4}\right)\cdot d\times\\
&\int\frac{d\gamma}{\gamma} = \int\left(-\frac{1}{\times} - \frac{1}{4}\right)\cdot d\times\\
&ln\gamma = -ln\times - \frac{1}{4}\times + C_1\\
&ln\gamma = -ln\times + \ln e^{-\frac{\times}{4}} + \ln C_1\\
&ln\gamma = \frac{ln\, C_1\cdot e^{-\frac{\times}{4}}}{\times}\\
&\gamma = C_1\cdot\frac{e^{-\frac{\times}{4}}}{\times}\\
&\frac{dp}{d\times} = C_1\cdot\frac{e^{-\frac{\times}{4}}}{\times}
\end{aligned} \tag{20}$$

Based on Eq. 20, performing the variable separation and integrating, then the following expression represented by Eq. 21 is carried out:

$$dp = C_1\cdot\frac{e^{-\frac{\times}{4}}}{\times}\cdot d\times$$

By integrating both sides of the pressure equation, we simply get: (21)

$$\int dp = C_1\cdot\int\frac{e^{-\frac{\times}{4}}}{\times}\cdot d\times$$

$$P = C_1\cdot\int\frac{\aleph\cdot t\cdot e^{-\frac{\times}{4}}}{\times^2}\cdot d\times + C_2$$

For $t = 0$; $\times \to \infty$ the term within the integral goes to zero, so we will have the following expression in Eq.22:

$$\int \frac{\aleph \cdot t \cdot e^{-\frac{\times}{4}}}{\times^2} \cdot d\times = 0$$
$$\to for\ t = 0$$
$$\to P = P_i$$
$$P = C_2 = P_i \quad (22)$$

Starting from continuity Eq. 23 and substitute it further we get expression in Eq. 24:

$$\dot{Q} = v \cdot F \quad (23)$$

Hence, $\dot{Q} = 2\pi \cdot r \cdot h \cdot \frac{k}{\mu} \cdot \frac{\partial P}{\partial r}$

$$\dot{Q} = 2\pi \cdot r \cdot h \cdot \frac{k}{\mu} \cdot \frac{\partial P}{\partial \times} \cdot \frac{\partial \times}{\partial r}$$
$$\dot{Q} = 2\pi \cdot r \cdot h \cdot \frac{k}{\mu} \cdot \frac{\partial}{\partial r}\left(\frac{r^2}{\aleph \cdot t}\right) \cdot \frac{\partial P}{\partial \times}$$
$$\dot{Q} = 2\pi \cdot r \cdot h \cdot \frac{k}{\mu} \cdot \frac{2r}{\aleph \cdot t} \cdot \frac{dp}{d\times}$$
$$\dot{Q} = \frac{4\pi \cdot k \cdot h}{\mu} \cdot \frac{r^2}{\aleph \cdot t} \cdot \frac{dp}{d\times} \quad (24)$$
$$\dot{Q} = \frac{4\pi \cdot k \cdot h}{\mu} \cdot \times \cdot \frac{e^{-\frac{\times}{4}}}{\times} \cdot C_1$$
$$\dot{Q} = \frac{4\pi \cdot k \cdot h}{\mu} \cdot e^{-\frac{\times}{4}} \cdot C_1$$

For $t \to \infty \to e^{-\frac{\times}{4}} = 1$ than we can get the flow rate as given in expression in Eq.25.

$$\dot{Q} = \frac{4\pi \cdot k \cdot h}{\mu} \cdot C_1 \cdot e^{-\frac{r^2}{4 * \aleph * t}}$$
$$\to e^{-\frac{r^2}{4 \cdot \aleph \cdot t}} = 1 \to C_1 = \frac{Q \cdot \mu}{4\pi \cdot k \cdot h} \quad (25)$$

As a conclusion the mathematical expression of the pressure as a function of two variables as chosen in the study, lead to the following expression in Eq.26:

$$P(r,t) = P_i + \frac{\dot{Q} \cdot \mu}{4\pi \cdot k \cdot h} \cdot \int \frac{e^{-\frac{\times}{4}}}{\times} \cdot d\times \quad (26)$$

Integrating we can get expressions as following in Eq.27

$$\int \frac{e^{-\frac{\times}{4}}}{\times} * d\times = \int \frac{e^{-\frac{\times}{4}}}{\frac{\times}{4}} \cdot d\left(-\frac{\times}{4}\right)$$
$$\int \frac{e^{-\frac{\times}{4}}}{\times} \cdot d\times = \int \frac{e^{-u}}{-u} \cdot du = E_i(-u)$$
$$P(r,t) = P_i - \frac{\dot{Q} \cdot \mu}{4\pi \cdot k \cdot h} \cdot E_i(-u) \quad (27)$$
$$E_i(-u) = \ln\frac{1}{u} - Ce$$
$$E_i(-u) = \ln\frac{1}{\frac{r^2}{4 \cdot \aleph \cdot t}} - Ce$$

Substituting $Ce = 0.5772$ (Euler constant) ($E_i(-u) = \ln\frac{4 \cdot \aleph \cdot t}{r^2} - 0.5772$, as well as $r = r_w$. Assuming that our analysis and pressure values are directly measured in the wellbore, and further substituting the above values, then the mathematical expression given in Eq. 28 regarding diffusivity equation is carried out.

$$P(r,t) = P_i - \frac{Q \cdot \mu}{4\pi \cdot k \cdot h} \cdot \left(\ln\frac{4 \cdot \aleph \cdot t}{{r_w}^2} - 0.5772\right) \quad (28)$$

On the other hand, substituting $\aleph = \frac{k}{\emptyset \cdot \mu \cdot C}$ in Eq. 28 then the proposed mathematical model can be carried out and represented by the following Eq.29:

$$P(r,t) = P_i - \frac{Q \cdot \mu}{4\pi \cdot k \cdot h} \cdot \left(\ln\frac{4 \cdot k \cdot t}{\emptyset \cdot \mu \cdot C \cdot {r_w}^2} - 0.5772\right) \quad (29)$$

If more transformations are performed, then Eq.29 can be easily represented by the mathematical expression in Eq.30.

$$P(r,t) = = P_i - \frac{Q \cdot \mu}{4\pi \cdot k \cdot h} \cdot \left(\ln\frac{4 \cdot k \cdot t}{e^{0.5772} \cdot \emptyset \cdot \mu \cdot C \cdot {r_w}^2}\right) \quad (30)$$

The three forms of equations 28, 29, 30 are called the basis of the diffusion equation, since using dimensionless variables such as dimensionless radius, dimensionless time and dimensionless pressure, for a many reasons, as well as using conversion to field unit and mechanical skin factor [6], the equation's form will be change, [1], [4]. As can be seen, the transient solution is not true for the entire drainage surface, as the reservoir appears to be infinite in extent, at the moment we refer the position of the well in relation to the contour, for a short time when the equation is applicable, [1], [4]. The author in the study [8] suggested a method for determination of average pressure in a bounded reservoir. resolves the question on issues related to drainage volumes with no pressure data. In that case a plot of average pressure versus relative drainage volume may allow the missing pressures to be assessed, [8]. In the research work of [21] the flow of a fluid with pressure-dependent viscosity through variable permeability porous layer is performed. The results showed that values of the permeability proportionality constant have negligible or no effects on flow characteristics.

4 Conclusion

As conclusion, the diffusivity equation presented above, expresses a connection between the principles and laws of physics and further structured to mathematical analysis employing differential equations, coordinate transformations, derivation, and integration rules. Diffusion itself represents a physical phenomenon of molecular movements, usually manifested by the movement of liquids and gases depending on the conditions and parameters impacting it. For the case study, applicable to oil and gas-bearing rocks, the pressure diffusion in the reservoir, is mainly affected by the Darcy law (filtration velocity of fluids in porous media), the law of mass conservation, the equation of state for fluid and rock structure as well. The study of the filtration process of slightly compressible liquids in porous media as well as the determination of its dynamics pressure drop, and rates of exploitation time for a given oil field, interconnected, and combined in the diffusion equation, have a great importance in practical applications during well testing. In this research paper, the radial flow is treated in a layer to an infinite extent exploited with a constant flow rate. Further on it is mathematically represented by solving the differential equation gathering different variables we suggest the new mathematical technique useful in both the theoretical and practical aspects during well testing. The proposed method is simple and applicable in the real conditions of fluid flow in porous media and regardless of certain limitations that exist during the solution of the diffusion equation, all the mathematical relationships of the different variables expressed above, which are programmed in the corresponding software, not only provide a quick and concise solution but help in many situations in the calculation of various parameters during the hydrodynamic study of wells.

5 Future Work

In the future, all these parameters taken into consideration and their mathematical relationship expressed based on the physical concept of fluid mobility in the porous medium make it possible not only to solve the diffusion equation in a different and simple mathematical method but also the variables that take parts in this equation, which are expressed as a function of different variables, help to solve many problems encountered in the testing and hydrodynamic analysis of wells, leading to improve the fuel extraction economy and fuel quality for a better and safer environmental especially from transport sector, [22], [23].

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The authors have no conflicts of interest to declare.