A different perspective on undeniable roles of some well-defined special functions in various multidisciplinary applications

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Abstract - In this special note, certain perspectives on various undeniable roles of certain special functions are considered. First, the selection of the relevant special function is discussed with a focus on the well-known gamma function. Then, essential information about this special function is provided, and important relationships between the selected function and the Caputo derivative(s) of fractional order are established. Finally, the possible effects of the relevant fractional derivative on certain types of special functions are revolved as computational and theoretical researches.

Key-Words: - Improper integrals, exponential forms, the error functions, the gamma function, various differential type equations.

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1 Introduction and preliminary information

The (classical) gamma function has always captured the attention of some of the most prominent mathematicians in history. Its history, as documented by Philip J. Davis in an article that won the 1963 Chauvenet Prize, reflects many major developments in mathematics since the 18th century. For further details, one may refer to [1]. Since then, the gamma function has continued to interest mathematicians in various ways. This special function, defined by an improper integral, is just one of many special functions in mathematical literature and has a wide range of applications in various branches of mathematics, including probability and statistics.

As is well known, this special function with the parameter r, which can be consisted of any real (or complex) numbers, is generally denoted by the notation $\Gamma(r)$ and is also defined as

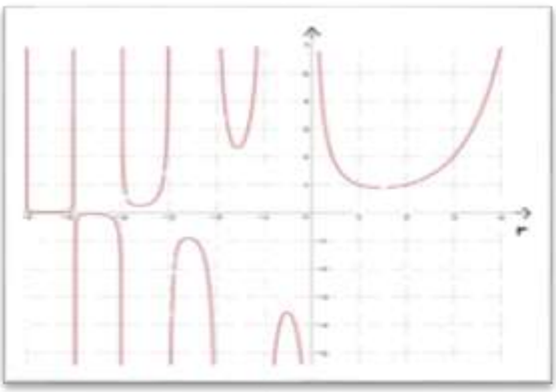
$$\Gamma(r) = \int_0^\infty Q^{r-1}e^{-Q}dQ, \qquad (1)$$

which, of course, must be convergent. According to classical analysis, the existence of the improper integral given by (1) is guaranteed only if the parameter r satisfies the following conditions:

$$r \in \mathbf{R} := \mathbb{R} - \{0, -1, -2, \cdots\},$$

where $\mathbb{R}$ denotes the set of real numbers.

In particular, for different selections of the relevant parameter (or variable) r ($r \in \mathbf{R}$), the following graph of the function $\Gamma(r)$, as defined by (1), can easily be generated using various computer mathematical programs, as shown in Graph 1. Naturally, this graph also provides us with diverse and valuable information.



The graph 1: The gamma function with real parameter r

In particular, when the parameter r is chosen as a natural number, which belongs to the familiar set

$$\mathbb{N} := \{1, 2, 3, \cdots\},$$

and the method of integration by parts is applied, an essential relationship between the improper integral given by (1) and the factorial can be easily observed, as follows:

$$\begin{aligned}\Gamma(r+1) &= \int_0^\infty Q^r e^{-Q}\, dQ \\ &= -Q^{r-1}e^{-Q}\big|_0^\infty \\ &\qquad + r\int_0^\infty Q^{r-1}e^{-Q}dQ \\ \Gamma(r+1) &= r\Gamma(r) \qquad (2) \\ &= \cdots \\ &= r! \qquad (3)\end{aligned}$$

for all $r \in \mathbf{N} := \mathbb{N} \cup \{0\}$.

In light of both our knowledge of classical analysis and the data shown in the Graph 1, since zero and negative integers are undefined points for the gamma function with a real parameter, the convergence of the improper integral in (1) is, of course, out of the question.

On the other hand, although the gamma function is defined for every number in the set $\mathbf{R}$ and the related integral is naturally convergent, calculating the values of $\Gamma(r)$ using elementary methods is not always straightforward. Nonetheless, we would like to highlight some of the various special properties:

- Using the fundamental assertion given in (3), it is easy to see that

  $$\Gamma(1) = 0!, \Gamma(2) = 1! \text{ and } \Gamma(3) = 2!,$$

  which are also required to the following improper integrals:

  $$\int_0^\infty Q^{1-1}e^{-Q}dQ = 1,$$

  $$\int_0^\infty Q^{2-1}e^{-Q}dQ = 1$$

  and

  $$\int_0^\infty Q^{3-1}e^{-Q}dQ = 2,$$

  respectively.

- The recursive relationship given by (2) holds for all positive real numbers r $(r > 0)$, not just integers. For example, it is well known that

  $$\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$$

  and

  $$\Gamma\left(\frac{3}{2}\right) = \frac{1}{2}\Gamma\left(\frac{1}{2}\right) = \frac{\sqrt{\pi}}{2},$$

which are also essential results for the following improper integrals:

$$\int_0^\infty Q^{-1/2}e^{-Q}dQ = \sqrt{\pi}$$

and

$$\int_0^\infty Q^{1/2}e^{-Q}dQ = \frac{\sqrt{\pi}}{2},$$

respectively.

- At the same time, the relationship takes the form given by

$$\Gamma(r) = \frac{\Gamma(r+1)}{r} \qquad (4)$$

gives us a suitable way to extend the gamma function to the negative real numbers. For example, the value of

$$\Gamma\left(-\frac{1}{2}\right)$$

could also be evaluated by the help of the relation in (4), which immediatley yields that

$$\Gamma\left(-\frac{1}{2}\right) = \frac{\Gamma\left(-\frac{1}{2}+1\right)}{-\frac{1}{2}}$$
$$= -2\Gamma\left(\frac{1}{2}\right)$$
$$= -2\sqrt{\pi}.$$

In light of the special examples above and using the relationships between (2) and (4), it is also possible to determine the values for each suitable rational-type real number in intervals such as

$$(2,3),\ (3,4),\ (5,6),\ \cdots$$

and

$$(-2,-1),\ (-3,-2),\ (-4,-3),\ \cdots,$$

it is also possible to determine their values. For example,

$$\Gamma\left(\frac{3}{2}\right) = ?,\ \Gamma\left(\frac{5}{2}\right) = ?,\ \Gamma\left(\frac{7}{2}\right) = ?,\ \ldots$$

and

$$\Gamma\left(-\frac{3}{2}\right) = ?,\ \Gamma\left(-\frac{5}{2}\right) = ?,\ \Gamma\left(-\frac{7}{2}\right) = ?,\ \ldots$$

In particular, within the scope of the Gamma function defined by (1), we leave the comparison of the results we have previously calculated with the data presented in the Graph 1 and the use of this graph to determine the locations corresponding to the additional results (values) mentioned above to the special interest of researchers.

For more detailed information about the special function, including its extensive properties and related implications (or possible applications), one can refer to the essential research found in the references [2], [3], [4], [5], [6], [7], [8], and [9].

2 The Gamma Function, Special Definitions, Various Implications and Examples, and Conclusions

In particular, focusing on the main topic of this scientific note, we aim to explore the concept of 'the function defined by (1)' as one of the relevant expressions of 'some well-defined functions' in our study. To this end, we will first provide the necessary information and then delve into some specific investigations and expressions.

When the r-th order (ordinary) derivative is considered for any continuous (*or* piecewise continuous) function such as

$$y = f(t) = t^s,$$

the existence of the result (*or* expression) of the familiar form:

$$\left(\frac{d}{dt}\right)^r (t^s) \equiv \frac{d^r}{dt^r}(t^s)$$
$$\equiv (t^s)^{(r)}$$
$$= s\,(s-1)\,(s-2)$$
$$\cdot \cdots \cdot \big(s-(r+1)\big)\, t^{s-r}$$
$$= \frac{s!}{(s-r)!} t^{s-r} \quad (r < s) \qquad (5)$$

is obvious to all of us, where

$$s \in \mathbb{N} \quad \text{and} \quad r \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}.$$

Furthermore, when the special definition of the function given in (1) is applied to the result (or expression) in (5), for a function $y = f(t)$, the expression in (5) can naturally be restated in the following form:

$$\left(\frac{d}{dt}\right)^\rho [f(t)] \equiv \frac{d^\rho}{dt^\rho}[f(t)]$$
$$= \left(\frac{d}{dt}\right)^\rho (t^s)$$
$$= \frac{\Gamma(s+1)}{\Gamma(s-\rho+1)} t^{s-\rho} \qquad (6)$$

as the expectable form relating to the fractional type derivative(s), where

$$s - 1 \le \rho < s \ \text{ and } \ s \in \mathbb{N}. \qquad (7)$$

As some simple examples, under the conditions specified in (7), and considering this new formulation along with the special information provided in (2) and (4) and focusing on the extensive assertion in (6), the special results are given by

$$s = 1 \Rightarrow \left(\frac{d}{dt}\right)^{1/2}(t) = \frac{\Gamma(2)}{\Gamma(3/2)} t^{1/2} = \frac{2}{\sqrt{\pi}}\sqrt{t} \qquad (8)$$

and

$$s = 1 \Rightarrow \left(\frac{d}{dt}\right)^{1/2}(t^2) = \frac{\Gamma(3)}{\Gamma(5/2)} t^{3/2} = \frac{2!}{\Gamma(5/2)} t\sqrt{t} = \frac{2}{3/2\,\Gamma(3/2)} t\sqrt{t} = \frac{8}{3\sqrt{\pi}} t\sqrt{t}$$

can easily be calculated as two more especial examples. In the same time, this expresses that the relevant-type derivatives of order $1/2$ of the functions being of

$$f(t) = t \quad \text{and} \quad f(t) = t^2$$

at the point $t_0 = 1$ also are equal to more special results given by

$$\left(\frac{d}{dt}\right)^{1/2}(t)\bigg|_{t:=1} = \frac{2}{\sqrt{\pi}}$$

and

$$\left(\frac{d}{dt}\right)^{1/2}(t)\bigg|_{t:=1} = \frac{8}{3\sqrt{\pi}},$$

respectively.

We specifically draw your attention to the fact that, although the ordinary derivatives of the elementary functions mentioned above at the point $t_0 = 1$ are $1/2 = 0.5$ and 2, their fractional derivatives of order $1/2$ at that point are $2/\sqrt{\pi} \approx 0.636953$ and $8/(3\sqrt{\pi}) \approx 0.849257$, respectively.

Although the definition established by (6) is a relatively simple approach, the extensive impact of the Gamma function, as defined in (1), has made significant contributions to both various applications and theoretical frameworks in the literature. It remains an active area of research. In such cases, the following chronological flow will provide specific information about diverse calculations (or operators).

The special result in (8) was first considered by Sylvestre F. Lacroix in 1819. Subsequently, in 1823, Niels Abel developed his theory of fractional order derivatives and integrals, and applied it practically to the Tautochron problem. This problem involves finding the equation of a curve along which a point mass rolls to the lowest point in the same amount of time under the influence of constant gravity, regardless of the starting point on the curve. Joseph Liouville [10] made a significant attempt to present a formal definition of a fractional derivative. In 1847, B. Riemann published a private paper posthumously, in which he provided a definition of the fractional order differential operator, likely influenced by Liouville's results [11].

Specifically, those fractional-order derivatives are named in honor of Riemann and Liouville as the Riemann-Liouville fractional derivative and the Grünwald-Letnikov derivative. Formal definitions of these derivatives can be found in sources such as [12], [13], [14], and [15]. In 1967, M. Caputo, while solving certain boundary value problems in the theory of viscoelasticity, formulated a new definition of the fractional-order derivative [16]. The main advantage of the Caputo approach is that the initial and boundary conditions for differential equations involving the Caputo fractional derivative are analogous to those for integer-order differential equations, allowing for similar interpretations. Consequently, it is frequently used in various practical applications. In 2000, R. Hilfer proposed a new definition of the fractional-order derivative. The two-parameter family of Hilfer fractional derivatives generalizes both the Caputo derivative and the Riemann-Liouville derivative, allowing interpolation between these two types of fractional derivatives. For further details, see [17] and [18]. Additionally, earlier papers on related topics can be found in [19], [20], and [21].

In short, under the conditions in (7), by considering the expressions given by (1) and (6) together, the fractional-order-type computations, namely the relevant operators emphasized (just) above can also focused on. We will present the details of related studies for the attention of interested researchers and focus specifically on the Caputo fractional-order derivative. Its familiar definition is provided below.

As an especial definition, for a given function such as

$$f := f(t) : (0,\infty) \to \mathbb{R},$$

the Caputo derivative of fractional-order ρ (*or* the Caputo derivative of order ρ) is denoted by

$${}^{C}\mathfrak{D}^{\rho}[f(t)] \equiv {}^{C}\mathfrak{D}^{\rho}[f]$$

and defined by

$${}^{C}\mathfrak{D}^{\rho}[f(t)] = \frac{1}{\Gamma(s-\rho)}\int_0^{v} \frac{f^{(s)}(w)}{(v-w)^{\rho-s+1}}dw, \qquad (9)$$

where $s-1 \le \rho < s$ $(s \in \mathbb{N})$ and, *of course*, the significant $f^{(s)}(w)$ $(s \in \mathbb{N}_0)$ denotes s-th (ordinary) derivative (with respect to the independent variable w) of the function f given.

We should particularly note that if the integral given by (9) exists, then the Caputo derivative of the relevant function will also exist.

Additionally, the fractional-order Caputo operator is a well-defined operator. From a basic perspective, two fundamental properties of this operator, which are scalar multiplication and linearity, are readily apparent.

At the same time, if the parameter ρ approaches the parameter s, it is evident that the ordinary derivative of any continuous (or piecewise continuous) function f can also be obtained. A detailed examination of such special relationships and results is left to the dedicated efforts of researchers interested in these areas.

There are certainly numerous applications of the relevant operator across various fields of science and engineering. In particular, in mathematics, its applications are extensive, especially within the context of special functions. We will focus on just one such application within the scope of the definition given by (9).

Traditionally, the differential equation presented below is a special equation that is directly related to many real-world problems. One such problem is foundational to nuclear physics. From this perspective, when considering the special function of the form:

$$\mathcal{N}(\tau) = \mathcal{N}_0 e^{-\lambda\tau}, \qquad (10)$$

which is a solution of the following first-order-linear-ordinary differential equation given by

$$\frac{d\mathcal{N}}{d\tau} = -\lambda\mathcal{N} \ \left(\mathcal{N} := \mathcal{N}(\tau)\right). \qquad (11)$$

As it is known, the significant equation (above) describes radioactive decay, where $\mathcal{N}$ denotes the number of remaining radioactive atoms in a sample after time τ, and λ is the decay constant. For the extensive applications of expressions (10) and (11), it would be useful to consult the essential works listed in [2], [3], [4], [7], [8], and [14].

We would like to conclude this investigation by examining the function given by (10) in the context of the operator defined in (9), as detailed in the implication below.

As some implications, under the conditions mentioned in (7), and in light of the information provided in (6), using the definition given in (9) and considering the Maclaurin series expansion:

$$e^{w} = \sum_{j=0}^{\infty}\frac{1}{j!}w^{j} \quad (w \in \mathbb{R}),$$

and after some elementary calculations, we then get that

$${}^{C}\mathfrak{D}^{\rho}\left[e^{-\lambda\tau}\right] = \frac{1}{\Gamma(s-\rho)}\int_0^{v}\frac{\left(\mathcal{N}_0 e^{-\lambda w}\right)^{(s)}}{(v-w)^{\rho-s+1}}dw$$

$$= \frac{\mathcal{N}_0(-\lambda)^{s}}{\Gamma(s-\rho)}\int_0^{v}\left(\frac{\sum_{j=0}^{\infty}\frac{(-\lambda)^{j}}{j!}w^{j}}{(v-w)^{\rho-s+1}}\right)dw$$

$$= \mathcal{N}_0(-\lambda)^{s} \ \sum_{j=0}^{\infty}\frac{(-\lambda)^{j}\,\tau^{j+s-\rho}}{\Gamma(j+s-\rho+1)},$$

or, equivalently,

$${}^{C}\mathfrak{D}^{\rho}\left[e^{-\lambda\tau}\right] = (-\lambda)^{s}\,\tau^{s-\rho}\,\mathrm{E}_{1,s-\rho+1}(-\lambda\tau),$$

where the notation $\mathbf{E}_{u,v}(t)$ is called as the two-parameters Mittag-Leffler function and it is also defined by

$$\mathbf{E}_{u,v}(w) = \sum_{j=0}^{\infty}\frac{w^{j}}{\Gamma(uw+v)} \quad (u,v>0; w \in \mathbb{R}),$$

which is its Caputo derivative of order ρ of the exponential type function being of the form given by (10).

In addition, as an extra special result of our implications, in the light of the Implication (just above), when setting $\rho := 1/2$ and $\lambda := -1$ there, the special example, consisting of the Caputo derivative of order 1/2, which belongs to the exponential function $f(t) = e^t$, can also be obtained in the equivalent forms given by

$$
\begin{aligned}
{}^{C}\mathfrak{D}^{1/2}\,[e^{\tau}] &= \sum_{j=0}^{\infty} \frac{\tau^{j+1/2}}{\Gamma(j+3/2)} \\
&= \sqrt{\tau}\,\mathbf{E}_{1,3/2}(\tau) \\
&= e^{\tau} erf\big(\sqrt{\tau}\big) \\
&= e^{\tau}\big[1 - erfc\big(\sqrt{\tau}\big)\big],
\end{aligned}
$$

where $\tau > 0$ and the special functions:

$$erf(\cdot) \quad \text{and} \quad erfc(\cdot)$$

are known as the *error* function and the *complementary error* function, respectively, and they are also defined by

$$erf(\tau) = \frac{2}{\sqrt{\pi}} \int_0^{\tau} e^{-\kappa^2}\, d\kappa$$

and

$$erfc(\tau) = 1 - \frac{2}{\sqrt{\pi}} \int_{\tau}^{\infty} e^{-\kappa^2}\, d\kappa\,,$$

where $\tau \in \mathbb{R}$.

In particular, many earlier investigations given in [4], [8], [22], [23], [24], [25] and [26] can be consulted for additional information about various type error functions and their comprehensive applications.

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