Growth estimates of solutions of linear differential equations with dominant coefficient of lower (α, β, γ)-order

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Abstract: - In this paper, we deal with the growth and oscillation of solutions of higher order linear differential equations. Under the conditions that there exists a coefficient which dominates the other coefficients by its lower (α, β, γ)-order and lower (α, β, γ)-type, we obtain some growth and oscillation properties of solutions of such equations which improve and extend some recently results of the author and Biswas [4].

Key-Words: - Differential equations, (α, β, γ)-order, lower (α, β, γ)-order, (α, β, γ)-type, lower (α, β, γ)-type, growth of solutions.

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1 Introduction

Throughout this paper, we assume that the reader is familiar with the fundamental results and the standard notations of the Nevanlinna value distribution theory of meromorphic functions [11, 15, 45].

Nevanlinna theory has appeared to be a powerful tool in the field of complex differential equations. For an introduction to the theory of differential equations in the complex plane by using the Nevanlinna theory see [25]. Active research in this field was started by Wittich [42, 43] and his students in the 1950's and 1960's. After their many authors have investigated the complex differential equations

$$f^{(k)}(z) + A_{k-1}(z)f^{(k-1)}(z) + \cdots + A_1(z)f'(z) + A_0(z)f(z) = 0, \tag{1}$$

$$f^{(k)}(z) + A_{k-1}(z)f^{(k-1)}(z) + \cdots + A_1(z)f'(z) + A_0(z)f(z) = F, \tag{2}$$

and achieved many valuable results when the coefficients $A_0(z), ..., A_{k-1}(z)$, $(k \geq 2)$ and $F(z)$ in (1) and (2) are entire or meromorphic functions of finite order or finite iterated p-order or (p, q)-th order or (p, q)-φ order; see ([5], [8], [14], [18], [23], [25], [27], [28], [29], [37], [39], [40], [44]).

Chyzhykov and Semochko [9] showed that both definitions of iterated p-order ([20], [23], [34], [35]) and the (p, q)-th order ([21], [22]) have the disadvantage that they do not cover arbitrary growth (see [9, Example 1.4]). They used more general scale, called the φ-order (see [9], [36]) and the concept of φ-order is used to study the growth of solutions of complex differential equations in the whole complex plane and in the unit disc which extend and improve many previous results see ([1, 9, 36]). Extending this notion, Long et al. [30] recently introduce the concepts of $[p, q]_{,\varphi}$-order and $[p, q]_{,\varphi}$-type (see [30]) and obtain some interesting results which considerably extend and improve some earlier results. For details one may see [30].

The concept of generalized order (α, β) of an entire function was introduced by Sheremeta [38]. Several authors made close investigations on the properties of entire functions related to generalized order (α, β) in some different direction [6, 7]. On the other hand, Mulyava et al. [31] have used the concept of (α, β)-order of an entire function in order to investigate the properties of solutions of a heterogeneous differential equation of the second order and obtained several remarkable results. For details about (α, β)-order one may see [31, 38].

Now, let L be a class of continuous non-negative on $(-\infty, +\infty)$ function α such that $\alpha(x) = \alpha(x_0) \geq 0$ for $x \leq x_0$ and $\alpha(x) \uparrow +\infty$ as $x_0 \leq x \to +\infty$. We say that $\alpha \in L_1$, if $\alpha \in L$ and $\alpha(a + b) \leq \alpha(a) + \alpha(b) + c$ for all $a, b \geq R_0$ and fixed $c \in (0, +\infty)$. Further, we say that $\alpha \in L_2$, if $\alpha \in L$ and $\alpha(x + O(1)) = (1 + o(1))\alpha(x)$ as $x \to +\infty$. Finally, $\alpha \in L_3$, if $\alpha \in L$ and $\alpha(a + b) \leq \alpha(a) + \alpha(b)$ for all $a, b \geq R_0$, i.e., α is subadditive. Clearly $L_3 \subset L_1$.

Particularly, when $\alpha \in L_3$, then one can easily verify that $\alpha(mr) \leq m\alpha(r)$, $m \geq 2$ is an integer. Up to a normalization, subadditivity is implied by concavity. Indeed, if $\alpha(r)$ is concave on $[0, +\infty)$ and satisfies $\alpha(0) \geq 0$, then for $t \in [0,1]$,

$$\begin{aligned}\alpha(tx) = \alpha(tx + (1-t)\cdot 0) &\geq t\alpha(x) + (1-t)\alpha(0) \\ &\geq t\alpha(x),\end{aligned}$$

so that by choosing $t = \frac{a}{a+b}$ or $t = \frac{b}{a+b}$,

$$\begin{aligned}\alpha(a+b) &= \frac{a}{a+b}\alpha(a+b) + \frac{b}{a+b}\alpha(a+b) \\ &\leq \alpha\left(\frac{a}{a+b}(a+b)\right) \\ &\quad + \alpha\left(\frac{b}{a+b}(a+b)\right) \\ &= \alpha(a) + \alpha(b), \quad a, b \geq 0.\end{aligned}$$

As a non-decreasing, subadditive and unbounded function, $\alpha(r)$ satisfies

$$\alpha(r) \leq \alpha(r + R_0) \leq \alpha(r) + \alpha(R_0)$$

for any $R_0 \geq 0$. This yields that $\alpha(r) \sim \alpha(r + R_0)$ as $r \to +\infty$.

Let α, β and γ satisfy the following two conditions : (i) Always $\alpha \in L_1$, $\beta \in L_2$ and $\gamma \in L_3$; and (ii) $\alpha(\log^{[p]} x) = o(\beta(\log\gamma(x)))$, $p \geq 2$, $\alpha(\log x) = o(\alpha(x))$ and $\alpha^{-1}(kx) = o\left(\alpha^{-1}(x)\right)$ $(0 < k < 1)$ as $x \to +\infty$.

Throughout this paper, we assume that α, β and γ always satisfy the above two conditions unless otherwise specifically stated.

Recently, Heittokangas et al. [19] have introduced a new concept of φ-order of entire and meromorphic functions considering φ as subadditive function. For details one may see [19]. Extending this notion, recently the author and Biswas [2] introduce the definition of the (α,β,γ)-order of a meromorphic function.

The main aim of this paper is to study the growth and oscillation of solutions of higher order linear differential equations using the concepts of lower (α,β,γ)-order and lower (α,β,γ)-type. In fact, some works relating to study the growth of solutions of higher order linear differential equations using the concepts of (α,β,γ)-order have been explored in [2], [3] and [4]. In this paper, we obtain some results which improve and generalize some previous results of the author and Biswas [4].

For $x \in [0,+\infty)$ and $k \in \mathbb{N}$ where $\mathbb{N}$ is the set of all positive integers, define iterations of the exponential and logarithmic functions as $\exp^{[k]} x = \exp(\exp^{[k-1]} x)$ and $\log^{[k]} x = \log(\log^{[k-1]} x)$ with convention that $\log^{[0]} x = x$, $\log^{[-1]} x = \exp x$, $\exp^{[0]} x = x$ and $\exp^{[-1]} x = \log x$.

Definition 1.1. *([2]) The (α,β,γ)-order denoted by $\rho_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f is defined by*

$$\rho_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha\left(\log T\left(r,f\right)\right)}{\beta\left(\log\gamma\left(r\right)\right)},$$

and for an entire function f, we define

$$\begin{aligned}\rho_{(\alpha,\beta,\gamma)}[f] &= \limsup_{r\to+\infty} \frac{\alpha\left(\log T\left(r,f\right)\right)}{\beta\left(\log\gamma\left(r\right)\right)} \\ &= \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} M(r,f))}{\beta\left(\log\gamma\left(r\right)\right)}.\end{aligned}$$

Similar to Definition 1.1, one can also define the lower (α,β,γ)-order of a meromorphic function f in the following way:

Definition 1.2. *The lower (α,β,γ)-order denoted by $\mu_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f is defined by*

$$\mu_{(\alpha,\beta,\gamma)}[f] = \liminf_{r\to+\infty} \frac{\alpha\left(\log T\left(r,f\right)\right)}{\beta\left(\log\gamma\left(r\right)\right)},$$

for an entire function f, one can easily by Theorem 7.1 in [11] verify that

$$\begin{aligned}\mu_{(\alpha,\beta,\gamma)}[f] &= \liminf_{r\to+\infty} \frac{\alpha\left(\log T\left(r,f\right)\right)}{\beta\left(\log\gamma\left(r\right)\right)} \\ &= \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]} M(r,f))}{\beta\left(\log\gamma\left(r\right)\right)}.\end{aligned}$$

Proposition 1.3. *([2]) If f is an entire function, then*

$$\begin{aligned}\rho_{(\alpha(\log),\beta,\gamma)}[f] &= \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} T(r,f))}{\beta\left(\log\gamma\left(r\right)\right)} \\ &= \limsup_{r\to+\infty} \frac{\alpha(\log^{[3]} M(r,f))}{\beta\left(\log\gamma\left(r\right)\right)},\end{aligned}$$

and also by Theorem 7.1 in [11], one can easily verify that

$$\begin{aligned}\mu_{(\alpha(\log),\beta,\gamma)}[f] &= \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]} T(r,f))}{\beta\left(\log\gamma\left(r\right)\right)} \\ &= \liminf_{r\to+\infty} \frac{\alpha(\log^{[3]} M(r,f))}{\beta\left(\log\gamma\left(r\right)\right)},\end{aligned}$$

where $(\alpha(\log),\beta,\gamma)$-order denoted by $\rho_{(\alpha(\log),\beta,\gamma)}[f]$ and lower $(\alpha(\log),\beta,\gamma)$-order denoted by $\mu_{(\alpha(\log),\beta,\gamma)}[f]$.

Now to compare the relative growth of two meromorphic functions having same non zero finite (α,β,γ)-order or non zero finite lower (α,β,γ)-order, one may introduce the definitions of (α,β,γ)-type and lower (α,β,γ)-type in the following manner:

Definition 1.4. *([4]) The (α,β,γ)-type denoted by $\tau_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f with $0 < \rho_{(\alpha,\beta,\gamma)}[f] < +\infty$ is defined by*

$$\tau_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\exp(\alpha(\log T(r,f)))}{(\exp(\beta(\log\gamma(r))))^{\rho_{(\alpha,\beta,\gamma)}[f]}}.$$

If f is an entire function with $\rho_{(\alpha,\beta,\gamma)}[f] \in (0,+\infty)$, then the (α,β,γ)-type of f is defined by

$$\tau_{(\alpha,\beta,\gamma),M}[f] = \limsup_{r\to+\infty} \frac{\exp(\alpha(\log^{[2]} M(r,f)))}{(\exp(\beta(\log\gamma(r))))^{\rho_{(\alpha,\beta,\gamma)}[f]}}.$$

Definition 1.5. *The lower (α,β,γ)-type denoted by $\underline{\tau}_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f with $0 < \mu_{(\alpha,\beta,\gamma)}[f] < +\infty$ is defined by*

$$\underline{\tau}_{(\alpha,\beta,\gamma)}[f] = \liminf_{r\to+\infty} \frac{\exp(\alpha(\log T(r,f)))}{(\exp(\beta(\log\gamma(r))))^{\mu_{(\alpha,\beta,\gamma)}[f]}}.$$

If f is an entire function with $\mu_{(\alpha,\beta,\gamma)}[f] \in (0,+\infty)$, then the lower (α,β,γ)-type of f is defined by

$$\underline{\tau}_{(\alpha,\beta,\gamma),M}[f] = \liminf_{r\to+\infty} \frac{\exp(\alpha(\log^{[2]} M(r,f)))}{(\exp(\beta(\log\gamma(r))))^{\mu_{(\alpha,\beta,\gamma)}[f]}}.$$

In order to study the oscillation properties of solutions of (1) and (2), we define the (α,β,γ)-exponent convergence of the zero-sequence of a meromorphic function f in the following way:

Definition 1.6. *([2]) The (α,β,γ)-exponent convergence of the zero-sequence denoted by $\lambda_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f is defined by*

$$\lambda_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log n(r,1/f))}{\beta(\log\gamma(r))} = \limsup_{r\to+\infty} \frac{\alpha(\log N(r,1/f))}{\beta(\log\gamma(r))}.$$

Analogously, the (α,β,γ)-exponent convergence of the distinct zero-sequence denoted by $\overline{\lambda}_{(\alpha,\beta,\gamma)}[f]$ of f is defined by

$$\overline{\lambda}_{(\alpha,\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log \overline{n}(r,1/f))}{\beta(\log\gamma(r))} = \limsup_{r\to+\infty} \frac{\alpha(\log \overline{N}(r,1/f))}{\beta(\log\gamma(r))}.$$

Accordingly, the values

$$\lambda_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} n(r,1/f))}{\beta(\log\gamma(r))} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} N(r,1/f))}{\beta(\log\gamma(r))}$$

and

$$\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f] = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \overline{n}(r,1/f))}{\beta(\log\gamma(r))} = \limsup_{r\to+\infty} \frac{\alpha(\log^{[2]} \overline{N}(r,1/f))}{\beta(\log\gamma(r))}$$

are respectively called as $(\alpha(\log),\beta,\gamma)$-exponent convergence of the zero-sequence and $(\alpha(\log),\beta,\gamma)$-exponent convergence of the distinct zero-sequence of a meromorphic function f.

Similar to Definition 1.6, one can also define the lower (α,β,γ)-exponent convergence of the zero-sequence of a meromorphic function f in the following way:

Definition 1.7. *The lower (α,β,γ)-exponent convergence of the zero-sequence denoted by $\underline{\lambda}_{(\alpha,\beta,\gamma)}[f]$ of a meromorphic function f is defined by*

$$\underline{\lambda}_{(\alpha,\beta,\gamma)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\log n(r,1/f))}{\beta(\log\gamma(r))} = \liminf_{r\to+\infty} \frac{\alpha(\log N(r,1/f))}{\beta(\log\gamma(r))}.$$

Analogously, the lower (α,β,γ)-exponent convergence of the distinct zero-sequence denoted by $\underline{\overline{\lambda}}_{(\alpha,\beta,\gamma)}[f]$ of f is defined by

$$\underline{\overline{\lambda}}_{(\alpha,\beta,\gamma)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\log \overline{n}(r,1/f))}{\beta(\log\gamma(r))} = \liminf_{r\to+\infty} \frac{\alpha(\log \overline{N}(r,1/f))}{\beta(\log\gamma(r))}.$$

Accordingly, the values

$$\underline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]} n(r,1/f))}{\beta(\log\gamma(r))} = \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]} N(r,1/f))}{\beta(\log\gamma(r))}$$

and

$$\underline{\overline{\lambda}}_{(\alpha(\log),\beta,\gamma)}[f] = \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]} \overline{n}(r,1/f))}{\beta(\log\gamma(r))} = \liminf_{r\to+\infty} \frac{\alpha(\log^{[2]} \overline{N}(r,1/f))}{\beta(\log\gamma(r))}$$

are respectively called as lower $(\alpha(\log),\beta,\gamma)$-exponent convergence of the zero-sequence and lower $(\alpha(\log),\beta,\gamma)$-exponent convergence of the distinct zero-sequence of a meromorphic function f.

Proposition 1.8. *([2]) Let $f_1(z)$, $f_2(z)$ be nonconstant meromorphic functions with $\rho_{(\alpha(\log),\beta,\gamma)}[f_1]$ and $\rho_{(\alpha(\log),\beta,\gamma)}[f_2]$ as their $(\alpha(\log),\beta,\gamma)$-order. Then*
(i) $\rho_{(\alpha(\log),\beta,\gamma)}[f_1 \pm f_2] \leq \max\{\rho_{(\alpha(\log),\beta,\gamma)}[f_1], \rho_{(\alpha(\log),\beta,\gamma)}[f_2]\}$;
(ii) $\rho_{(\alpha(\log),\beta,\gamma)}[f_2 \cdot f_2] \leq \max\{\rho_{(\alpha(\log),\beta,\gamma)}[f_1], \rho_{(\alpha(\log),\beta,\gamma)}[f_2]\}$;
(iii) If $\rho_{(\alpha(\log),\beta,\gamma)}[f_1] \neq \rho_{(\alpha(\log),\beta,\gamma)}[f_2]$, then

$$\rho_{(\alpha(\log),\beta,\gamma)}[f_1 \pm f_2] = \max\{\rho_{(\alpha(\log),\beta,\gamma)}[f_1], \rho_{(\alpha(\log),\beta,\gamma)}[f_2]\};$$

(iv) If $\rho_{(\alpha(\log),\beta,\gamma)}[f_1] \neq \rho_{(\alpha(\log),\beta,\gamma)}[f_2]$, then

$$\rho_{(\alpha(\log),\beta,\gamma)}[f_2 \cdot f_2] = \max\{\rho_{(\alpha(\log),\beta,\gamma)}[f_1], \rho_{(\alpha(\log),\beta,\gamma)}[f_2]\}.$$

By using the properties $T(r,f) = T(r,\frac{1}{f}) + O(1)$ and $T(r,af) = T(r,f) + O(1)$, $a \in \mathbb{C}\setminus\{0\}$, one can obtain the following result.

Proposition 1.9. *([4]) Let f be a non-constant meromorphic function. Then*
(i) $\rho_{(\alpha,\beta,\gamma)}[\frac{1}{f}] = \rho_{(\alpha,\beta,\gamma)}[f]$ $(f \not\equiv 0)$;
(ii) $\rho_{(\alpha(\log),\beta,\gamma)}[\frac{1}{f}] = \rho_{(\alpha(\log),\beta,\gamma)}[f]$ $(f \not\equiv 0)$;
(iii) If $a \in \mathbb{C}\setminus\{0\}$, then $\rho_{(\alpha,\beta,\gamma)}[af] = \rho_{(\alpha,\beta,\gamma)}[f]$ and $\tau_{(\alpha,\beta,\gamma)}[af] = \tau_{(\alpha,\beta,\gamma)}[f]$ if $0 < \rho_{(\alpha,\beta,\gamma)}[f] < +\infty$;
(iii) If $a \in \mathbb{C}\setminus\{0\}$, then $\rho_{(\alpha(\log),\beta,\gamma)}[af] = \rho_{(\alpha(\log),\beta,\gamma)}[f]$ and $\tau_{(\alpha(\log),\beta,\gamma)}[af] = \tau_{(\alpha(\log),\beta,\gamma)}[f]$ if $0 < \rho_{(\alpha(\log),\beta,\gamma)}[f] < +\infty$.

Proposition 1.10. *Let f, g be non-constant meromorphic functions with $\rho_{(\alpha(\log),\beta,\gamma)}[f]$ as $(\alpha(\log),\beta,\gamma)$-order and $\mu_{(\alpha(\log),\beta,\gamma)}[g]$ as lower $(\alpha(\log),\beta,\gamma)$-order. Then*

$$\mu_{(\alpha(\log),\beta,\gamma)}(f+g) \leq \max\left\{\rho_{(\alpha(\log),\beta,\gamma)}(f), \mu_{(\alpha(\log),\beta,\gamma)}(g)\right\}$$

and

$$\mu_{(\alpha(\log),\beta,\gamma)}(fg) \leq \max\left\{\rho_{(\alpha(\log),\beta,\gamma)}(f), \mu_{(\alpha(\log),\beta,\gamma)}(g)\right\}.$$

Furthermore, if $\mu_{(\alpha(\log),\beta,\gamma)}(g) > \rho_{(\alpha(\log),\beta,\gamma)}(f)$, then we obtain

$$\mu_{(\alpha(\log),\beta,\gamma)}(f+g) = \mu_{(\alpha(\log),\beta,\gamma)}(fg) = \mu_{(\alpha(\log),\beta,\gamma)}(g).$$

Proof. Without loss of generality, we assume that $\rho_{(\alpha(\log),\beta,\gamma)}(f) < +\infty$ and $\mu_{(\alpha(\log),\beta,\gamma)}(g) < +\infty$. From the definition of the lower $(\alpha(\log),\beta,\gamma)$-order, there exists a sequence $r_n \longrightarrow +\infty$ $(n \longrightarrow +\infty)$ such that

$$\lim_{n\longrightarrow+\infty} \frac{\alpha\left(\log^{[2]} T(r_n,g)\right)}{\beta(\log\gamma(r_n))} = \mu_{(\alpha(\log),\beta,\gamma)}(g).$$

Then, for any given $\varepsilon > 0$, there exists a positive integer N_1 such that

$$T(r_n,g) \leq \exp^{[2]}\{\alpha^{-1}\left(\left(\mu_{(\alpha(\log),\beta,\gamma)}(g) + \varepsilon\right) \times\beta(\log\gamma(r_n))\right)\}$$

holds for $n > N_1$. From the definition of the $(\alpha(\log),\beta,\gamma)-$order, for any given $\varepsilon > 0$, there exists a positive number R such that

$$T(r,f) \leq \exp^{[2]}\{\alpha^{-1}\left(\left(\rho_{(\alpha(\log),\beta,\gamma)}(f) + \varepsilon\right) \times\beta(\log\gamma(r))\right)\}$$

holds for $r \geq R$. Since $r_n \longrightarrow +\infty$ $(n \longrightarrow +\infty)$, there exists a positive integer N_2 such that $r_n > R$, and thus

$$T(r_n,f) \leq \exp^{[2]}\{\alpha^{-1}\left(\left(\rho_{(\alpha(\log),\beta,\gamma)}(f) + \varepsilon\right) \times\beta(\log\gamma(r_n))\right)\}$$

holds for $n > N_2$. Note that

$$T(r,f+g) \leq T(r,f) + T(r,g) + \ln 2$$

and

$$T(r,fg) \leq T(r,f) + T(r,g).$$

Then, for any given $\varepsilon > 0$, we have for $n > \max\{N_1, N_2\}$

$$\begin{aligned} T(r_n,f+g) &\leq T(r_n,f) + T(r_n,g) + \ln 2 \\ &\leq \exp^{[2]}\left\{\alpha^{-1}\left(\left(\rho_{(\alpha(\log),\beta,\gamma)}(f) + \varepsilon\right)\beta(\log\gamma(r_n))\right)\right\} \\ &+ \exp^{[2]}\{\alpha^{-1}\left(\left(\mu_{(\alpha(\log),\beta,\gamma)}(g) + \varepsilon\right) \times\beta(\log\gamma(r_n))\right)\} + \ln 2 \\ &\leq 3\exp^{[2]}\{\alpha^{-1}\left(\left(\max\left\{\rho_{(\alpha(\log),\beta,\gamma)}(f), \mu_{(\alpha(\log),\beta,\gamma)}(g)\right\} + \varepsilon\right)\beta(\log\gamma(r_n))\right)\} \end{aligned} \quad (3)$$

and

$$\begin{aligned} T(r_n,fg) &\leq T(r_n,f) + T(r_n,g) \\ &\leq 2\exp^{[2]}\{\alpha^{-1}\left(\left(\max\left\{\rho_{(\alpha(\log),\beta,\gamma)}(f), \mu_{(\alpha(\log),\beta,\gamma)}(g)\right\} + \varepsilon\right)\beta(\log\gamma(r_n))\right)\}. \end{aligned} \quad (4)$$

Since $\varepsilon > 0$ is arbitrary, then from (3) and (4), we easily obtain

$$\mu_{(\alpha(\log),\beta,\gamma)}(f+g) \leq \max\left\{\rho_{(\alpha(\log),\beta,\gamma)}(f), \mu_{(\alpha(\log),\beta,\gamma)}(g)\right\} \quad (5)$$

and

$$\begin{aligned}&\mu_{(\alpha(\log),\beta,\gamma)}(fg)\\&\leq \max\left\{\rho_{(\alpha(\log),\beta,\gamma)}(f),\mu_{(\alpha(\log),\beta,\gamma)}(g)\right\}.\end{aligned} \tag{6}$$

Suppose now that $\mu_{(\alpha(\log),\beta,\gamma)}(g) > \rho_{(\alpha(\log),\beta,\gamma)}(f)$. Considering that

$$\begin{aligned}T(r,g) &= T(r,f+g-f)\\&\leq T(r,f+g)+T(r,f)+\ln 2\end{aligned} \tag{7}$$

and

$$T(r,g) = T\left(r,\frac{fg}{f}\right) \leq T(r,fg)+T\left(r,\frac{1}{f}\right)$$
$$= T(r,fg)+T(r,f)+O(1). \tag{8}$$

By (7), (8) and the same method as above we obtain that

$$\begin{aligned}&\mu_{(\alpha(\log),\beta,\gamma)}(g)\\&\leq \max\left\{\mu_{(\alpha(\log),\beta,\gamma)}(f+g),\rho_{(\alpha(\log),\beta,\gamma)}(f)\right\}\\&= \mu_{(\alpha(\log),\beta,\gamma)}(f+g)\end{aligned} \tag{9}$$

and

$$\begin{aligned}&\mu_{(\alpha(\log),\beta,\gamma)}(g)\\&\leq \max\left\{\mu_{(\alpha(\log),\beta,\gamma)}(fg),\rho_{(\alpha(\log),\beta,\gamma)}(f)\right\}\\&= \mu_{(\alpha(\log),\beta,\gamma)}(fg).\end{aligned} \tag{10}$$

By using (5) and (9) we obtain $\mu_{(\alpha(\log),\beta,\gamma)}(f+g) = \mu_{(\alpha(\log),\beta,\gamma)}(g)$ and by (6) and (10), we get $\mu_{(\alpha(\log),\beta,\gamma)}(fg) = \mu_{(\alpha(\log),\beta,\gamma)}(g)$. □

2 Main Results

Very recently the author and Biswas have investigated the growth of solutions of equation (1) and established the following two results.

Theorem 2.1. *([4]) Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be entire functions such that $\rho_{(\alpha,\beta,\gamma)}[A_0] > \max\{\rho_{(\alpha,\beta,\gamma)}[A_j],\ j = 1, ..., k-1\}$. Then every solution $f(z) \not\equiv 0$ of (1) satisfies $\rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha,\beta,\gamma)}[A_0]$.*

Theorem 2.2. *([4]) Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be entire functions. Assume that*

$$\begin{aligned}&\max\{\rho_{(\alpha,\beta,\gamma)}[A_j], j = 1, ..., k-1\}\\&\leq \rho_{(\alpha,\beta,\gamma)}[A_0] = \rho_0 < +\infty\end{aligned}$$

and

$$\begin{aligned}&\max\{\tau_{(\alpha,\beta,\gamma),M}[A_j] : \rho_{(\alpha,\beta,\gamma)}[A_j] = \rho_{(\alpha,\beta,\gamma)}[A_0] > 0\}\\&< \tau_{(\alpha,\beta,\gamma),M}[A_0] = \tau_M.\end{aligned}$$

Then every solution $f(z) \not\equiv 0$ of (1) satisfies $\rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha,\beta,\gamma)}[A_0]$.

Theorems 2.1 and 2.2 concerned the growth properties of solutions of (1), when A_0 is dominating the others coefficients by its (α,β,γ)-order and (α,β,γ)-type. Thus, the natural question which arises: If A_0 is dominating coefficient with its lower (α,β,γ)-order and lower (α,β,γ)-type, what can we say about the growth of solutions of (1)? The following results give answer to this question.

Theorem 2.3. *Let $A_0(z), ..., A_{k-1}(z)$ be entire functions. Assume that $\max\{\rho_{(\alpha,\beta,\gamma)}[A_j] : j = 1, ..., k-1\} < \mu_{(\alpha,\beta,\gamma)}[A_0] \leq \rho_{(\alpha,\beta,\gamma)}[A_0] < +\infty$. Then every solution $f \not\equiv 0$ of (1) satisfies*

$$\underline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f-g] = \mu_{(\alpha,\beta,\gamma)}[A_0] = \mu_{(\alpha(\log),\beta,\gamma)}[f]$$

$$\leq \rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha,\beta,\gamma)}[A_0] = \overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f-g],$$

where $g \not\equiv 0$ is an entire function satisfying $\rho_{(\alpha(\log),\beta,\gamma)}[g] < \mu_{(\alpha,\beta,\gamma)}[A_0]$.

Theorem 2.4. *Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be entire functions. Assume that*

$$\max\{\rho_{(\alpha,\beta,\gamma)}[A_j] : j = 1, ..., k-1\} \leq \mu_{(\alpha,\beta,\gamma)}[A_0]$$
$$\leq \rho_{(\alpha,\beta,\gamma)}[A_0] = \rho < +\infty\ \ (0 < \rho < +\infty)$$

and

$$\begin{aligned}\tau_1 = \max\{\tau_{(\alpha,\beta,\gamma),M}[A_j] &: \rho_{(\alpha,\beta,\gamma)}[A_j]\\&= \mu_{(\alpha,\beta,\gamma)}[A_0] > 0\}\\&< \underline{\tau}_{(\alpha,\beta,\gamma),M}[A_0] = \tau\ \ (0 < \tau < +\infty).\end{aligned}$$

Then every solution $f \not\equiv 0$ of (1) satisfies

$$\underline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f-g] = \mu_{(\alpha,\beta,\gamma)}[A_0] = \mu_{(\alpha(\log),\beta,\gamma)}[f]$$

$$\leq \rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha,\beta,\gamma)}[A_0] = \overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f-g],$$

where $g \not\equiv 0$ is an entire function satisfying $\rho_{(\alpha(\log),\beta,\gamma)}[g] < \mu_{(\alpha,\beta,\gamma)}[A_0]$.

Theorem 2.5. *Let $A_0(z), ..., A_{k-1}(z)$ be entire functions. Assume that $\max\{\rho_{(\alpha,\beta,\gamma)}[A_j] : j = 1, ..., k-1\} \leq \mu_{(\alpha,\beta,\gamma)}[A_0] < +\infty$ and*

$$\limsup_{r\to+\infty} \frac{\sum_{j=1}^{k-1} m(r,A_j)}{m(r,A_0)} < 1.$$

Then every solution $f \not\equiv 0$ of (1) satisfies

$$\underline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f-g] = \mu_{(\alpha,\beta,\gamma)}[A_0] = \mu_{(\alpha(\log),\beta,\gamma)}[f]$$

$$\leq \rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha,\beta,\gamma)}[A_0] = \overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f-g],$$

where $g \not\equiv 0$ is an entire function satisfying $\rho_{(\alpha(\log),\beta,\gamma)}[g] < \mu_{(\alpha,\beta,\gamma)}[A_0]$.

Theorem 2.6. *Let $A_0(z), ..., A_{k-1}(z)$ be entire functions such that $A_0(z)$ is transcendental. Assume that* $\max\{\rho_{(\alpha,\beta,\gamma)}[A_j] : j = 1, ..., k-1\} \leq \mu_{(\alpha,\beta,\gamma)}[A_0] = \rho_{(\alpha,\beta,\gamma)}[A_0] < +\infty$ *and*

$$\liminf_{r\to+\infty} \frac{\sum_{j=1}^{k-1} m(r, A_j)}{m(r, A_0)} < 1, \ r \notin E,$$

where E is a set of r of finite linear measure. Then every solution $f \not\equiv 0$ of (1) satisfies

$$\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f-g] = \mu_{(\alpha,\beta,\gamma)}[A_0] = \mu_{(\alpha(\log),\beta,\gamma)}[f]$$

$$= \rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha,\beta,\gamma)}[A_0] = \overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f-g],$$

where $g \not\equiv 0$ is an entire function satisfying $\rho_{(\alpha(\log),\beta,\gamma)}[g] < \mu_{(\alpha,\beta,\gamma)}[A_0]$.

Remark 2.7. *Nevanlinna theory, originally part of complex analysis, has broad applications in both applied science and advanced mathematical methods. In signal processing and control theory, Nevanlinna theory helps analyze system stability and signal behavior by determining how often critical values are reached by a system. It is particularly useful in designing robust filtering systems and feedback controls, minimizing noise, and ensuring stability. In mathematical physics, Nevanlinna theory aids in understanding the behavior of complex systems like wave propagation, quantum mechanics, and electromagnetic fields. Nevanlinna theory also plays a crucial role in algebraic geometry and Diophantine approximation, where it helps study the distribution of rational points on algebraic varieties. Its connections to the Mordell conjecture (Faltings' theorem) show its relevance in the intersection of complex analysis with modern topology and algebraic methods. In both applied sciences and advanced mathematics, Nevanlinna theory provides a powerful tool for analyzing value distributions and system dynamics, please see, [10], [12], [24], [26], [32], [33].*

3 Preliminary Lemmas

In this section we present some lemmas which will be needed in the sequel. First, we denote the Lebesgue linear measure of a set $E \subset [0, +\infty)$ by $m(E) = \int_F dt$, and the logarithmic measure of a set $F \subset (1, +\infty)$ by $m_l(F) = \int_F \frac{dt}{t}$.

The following result due to Gundersen [13] plays an important role in the theory of complex differential equations.

Lemma 3.1. *([13]) Let f be a transcendental meromorphic function, and let $\chi > 1$ be a given constant. Then there exist a set $E_1 \subset (1, \infty)$ with finite logarithmic measure and a constant $B > 0$ that depends only on χ and i, j $(0 \leq i < j \leq k)$, such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have*

$$\left|\frac{f^{(j)}(z)}{f^{(i)}(z)}\right| \leq B \left\{\frac{T(\chi r, f)}{r} (\log^{\chi} r) \log T(\chi r, f)\right\}^{j-i}.$$

Lemma 3.2. *Let f be a meromorphic function with $\mu_{(\alpha(\log),\beta,\gamma)}[f] = \mu < +\infty$. Then there exists a set $E_2 \subset (1, +\infty)$ with infinite logarithmic measure such that for $r \in E_2 \subset (1, +\infty)$, we have for any given $\varepsilon > 0$*

$$T(r, f) < \exp^{[2]}\left\{\alpha^{-1}\left((\mu + \varepsilon)\beta(\log\gamma(r))\right)\right\}.$$

Proof. The definition of lower $(\alpha(\log), \beta, \gamma)$-order implies that there exists a sequence $\{r_n\}_{n=1}^{+\infty}$ tending to ∞ satisfying $\left(1 + \frac{1}{n}\right) r_n < r_{n+1}$ and

$$\lim_{r_n\to\infty} \frac{\alpha(\log^{[2]} T(r_n, f))}{\beta(\log\gamma(r_n))} = \mu_{(\alpha(\log),\beta,\gamma)}[f].$$

Then for any given $\varepsilon > 0$, there exists an integer n_1 such that for all $n \geq n_1$,

$$T(r_n, f) < \exp^{[2]}\left\{\alpha^{-1}\left(\left(\mu + \frac{\varepsilon}{2}\right)\beta(\log\gamma(r_n))\right)\right\}.$$

Set $E_2 = \bigcup_{n=n_1}^{+\infty} \left[\frac{n}{n+1} r_n, r_n\right]$. Then for $r \in E_2 \subset (1, +\infty)$, by using $\gamma(2r) \leq 2\gamma(r)$ and $\beta(r + O(1)) = (1 + o(1))\beta(r)$ as $r \to +\infty$, we obtain for any given $\varepsilon > 0$

$$\begin{aligned}
T(r, f) &\leq T(r_n, f) \\
&< \exp^{[2]}\left\{\alpha^{-1}\left(\left(\mu + \frac{\varepsilon}{2}\right)\beta(\log\gamma(r_n))\right)\right\} \\
&\leq \exp^{[2]}\left\{\alpha^{-1}\left(\left(\mu + \frac{\varepsilon}{2}\right) \times\beta\left(\log\gamma\left(\left(\frac{n+1}{n}\right)r\right)\right)\right)\right\} \\
&\leq \exp^{[2]}\left\{\alpha^{-1}\left(\left(\mu + \frac{\varepsilon}{2}\right)\beta(\log\gamma(2r))\right)\right\} \\
&\leq \exp^{[2]}\left\{\alpha^{-1}\left(\left(\mu + \frac{\varepsilon}{2}\right)\beta(\log(2\gamma(r)))\right)\right\} \\
&= \exp^{[2]}\left\{\alpha^{-1}\left(\left(\mu + \frac{\varepsilon}{2}\right)\beta(\log 2 + \log\gamma(r))\right)\right\} \\
&= \exp^{[2]}\left\{\alpha^{-1}\left(\left(\mu + \frac{\varepsilon}{2}\right)(1 + o(1))\beta(\log\gamma(r))\right)\right\} \\
&< \exp^{[2]}\left\{\alpha^{-1}\left((\mu + \varepsilon)\beta(\log\gamma(r))\right)\right\},
\end{aligned}$$

and $lm(E_2) = \sum_{n=n_1}^{+\infty} \int_{\frac{n}{n+1}r_n}^{r_n} \frac{dt}{t} = \sum_{n=n_1}^{+\infty} \log\left(1 + \frac{1}{n}\right) = \infty$. Thus, Lemma 3.2 is proved. □

We can also prove the following result by using similar reason as in the proof of Lemma 3.2.

Lemma 3.3. *Let f be an entire function with $\mu_{(\alpha,\beta,\gamma)}[f] = \mu < +\infty$. Then there exists a set $E_3 \subset (1,+\infty)$ with infinite logarithmic measure such that for $r \in E_3 \subset (1,+\infty)$, we have for any given $\varepsilon > 0$*

$$M(r,f) < \exp^{[2]}\left\{\alpha^{-1}\left((\mu+\varepsilon)\beta(\log\gamma(r))\right)\right\}.$$

The following lemma gives the relation between the maximum term and the central index of an entire function f.

Lemma 3.4. *([17], Theorems 1.9 and 1.10, or [20], Satz 4.3 and 4.4) Let $f(z) = \sum\limits_{n=0}^{+\infty} a_n z^n$ be an entire function, $\mu(r)$ be the maximum term of f, i.e.,*

$$\mu(r) = \max\{|a_n| r^n : n = 0,1,2,...\},$$

and $\nu(r,f) = \nu_f(r)$ be the central index of f, i.e.,

$$\nu(r,f) = \max\{m : \mu(r) = |a_m| r^m\}.$$

Then

(i)

$$\log\mu(r) = \log|a_0| + \int\limits_0^r \frac{\nu_f(t)}{t}dt,$$

here we assume that $|a_0| \neq 0$.

(ii) *For $r < R$*

$$M(r,f) < \mu(r)\left\{\nu_f(R) + \frac{R}{R-r}\right\}.$$

Lemma 3.5. *([16, 20, 41]) Let f be a transcendental entire function. Then there exists a set $E_4 \subset (1,+\infty)$ with finite logarithmic measure such that for all z satisfying $|z| = r \notin E_4$ and $|f(z)| = M(r,f)$, we have*

$$\frac{f^{(n)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^n (1+o(1)), \quad (n \in \mathbb{N}).$$

Here, we give the generalized logarithmic derivative estimates for meromorphic functions of finite $(\alpha(\log),\beta,\gamma)-$order.

Lemma 3.6. *([4]) Let f be a meromorphic function of order $\rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho < +\infty$, $k \in \mathbb{N}$. Then, for any given $\varepsilon > 0$,*

$$m\left(r,\frac{f^{(k)}}{f}\right) = O\left(\exp\left\{\alpha^{-1}((\rho+\varepsilon)\beta(\log\gamma(r)))\right\}\right),$$

outside, possibly, an exceptional set $E_5 \subset [0,+\infty)$ of finite linear measure.

Lemma 3.7. *([4]) Let $A_0(z), A_1(z), ..., A_{k-1}(z)$ be entire functions. Then every nontrivial solution f of* (1) *satisfies*

$$\rho_{(\alpha(\log),\beta,\gamma)}[f] \leq \max\{\rho_{(\alpha,\beta,\gamma)}[A_j] : j = 0,1,...,k-1\}.$$

Lemma 3.8. *([4]) Let f be an entire function with $\rho_{(\alpha,\beta,\gamma)}[f] = \rho \in (0,+\infty)$ and $\tau_{(\alpha,\beta,\gamma),M}[f] \in (0,+\infty)$. Then for any given $\eta < \tau_{(\alpha,\beta,\gamma),M}[f]$, there exists a set $E_6 \subset (1,+\infty)$ of infinite logarithmic measure such that for all $r \in E_6$, one has*

$$\exp\left\{\alpha(\log^{[2]} M(r,f))\right\} > \eta\left(\exp\{\beta(\log\gamma(r))\}\right)^\rho.$$

Lemma 3.9. *Let $f_2(z)$ be an entire function of lower $(\alpha(\log),\beta,\gamma)$-order with $\mu_{(\alpha(\log),\beta,\gamma)}[f_2] = \mu > 0$, and let $f_1(z)$ be an entire function of $(\alpha(\log),\beta,\gamma)$-order with $\rho_{(\alpha(\log),\beta,\gamma)}[f_1] = \rho < +\infty$. If $\rho_{(\alpha(\log),\beta,\gamma)}[f_1] < \mu_{(\alpha(\log),\beta,\gamma)}[f_2]$, then we have*

$$T(r,f_1) = o(T(r,f_2)) \text{ as } r \to +\infty.$$

Proof. By definitions of $(\alpha(\log),\beta,\gamma)$-order and lower $(\alpha(\log),\beta,\gamma)$-order, for any given ε with $0 < 2\varepsilon < \mu - \rho$ and sufficiently large r, we have

$$T(r,f_1) \leq \exp^{[2]}\left\{\alpha^{-1}\left((\rho+\varepsilon)\beta(\log\gamma(r))\right)\right\} \tag{11}$$

and

$$T(r,f_2) \geq \exp^{[2]}\left\{\alpha^{-1}\left((\mu-\varepsilon)\beta(\log\gamma(r))\right)\right\}. \tag{12}$$

Now by (11) and (12), we get

$$\frac{T(r,f_1)}{T(r,f)} \leq \frac{\exp^{[2]}\left\{\alpha^{-1}\left((\rho+\varepsilon)\beta(\log\gamma(r))\right)\right\}}{\exp^{[2]}\left\{\alpha^{-1}\left((\mu-\varepsilon)\beta(\log\gamma(r))\right)\right\}}$$

$$= \exp\{\exp\left\{\alpha^{-1}\left((\rho+\varepsilon)\beta(\log\gamma(r))\right)\right\} - \exp\left\{\alpha^{-1}\left((\mu-\varepsilon)\beta(\log\gamma(r))\right)\right\}\}$$

$$= \exp\left\{\left(\frac{\exp\left\{\alpha^{-1}\left((\rho+\varepsilon)\beta(\log\gamma(r))\right)\right\}}{\exp\left\{\alpha^{-1}\left((\mu-\varepsilon)\beta(\log\gamma(r))\right)\right\}} - 1\right) \times \exp\left\{\alpha^{-1}\left((\mu-\varepsilon)\beta(\log\gamma(r))\right)\right\}\right\}$$

$$= \exp\left\{\left(\frac{\exp\left\{\alpha^{-1}\left(\frac{\rho+\varepsilon}{\mu-\varepsilon}(\mu-\varepsilon)\beta(\log\gamma(r))\right)\right\}}{\exp\left\{\alpha^{-1}\left((\mu-\varepsilon)\beta(\log\gamma(r))\right)\right\}} - 1\right)\exp\left\{\alpha^{-1}\left((\mu-\varepsilon)\beta(\log\gamma(r))\right)\right\}\right\}.$$

Set

$$y = \left(\frac{\exp\left\{\alpha^{-1}\left(\frac{\rho+\varepsilon}{\mu-\varepsilon}\left(\mu-\varepsilon\right)\beta\left(\log\gamma\left(r\right)\right)\right)\right\}}{\exp\left\{\alpha^{-1}\left(\left(\mu-\varepsilon\right)\beta\left(\log\gamma\left(r\right)\right)\right)\right\}} - 1 \right) \times \exp\left\{\alpha^{-1}\left(\left(\mu-\varepsilon\right)\beta\left(\log\gamma\left(r\right)\right)\right)\right\}.$$

Then by putting $(\mu-\varepsilon)\beta(\log\gamma(r)) = x$, $\frac{\rho+\varepsilon}{\mu-\varepsilon} = k$ $(0<k<1)$ and making use of the condition $\alpha^{-1}(kx) = o\left(\alpha^{-1}(x)\right)$ $(0<k<1)$ as $x\to+\infty$, we get

$$\lim_{r\to+\infty} y = \lim_{x\to+\infty}\left(\frac{\exp\left\{\alpha^{-1}(kx)\right\}}{\exp\left\{\alpha^{-1}(x)\right\}} - 1\right)\exp\left\{\alpha^{-1}(x)\right\}$$

$$= \lim_{x\to+\infty}\left(\frac{\exp\left\{o\left(\alpha^{-1}(x)\right)\right\}}{\exp\left\{\alpha^{-1}(x)\right\}} - 1\right)\exp\left\{\alpha^{-1}(x)\right\}$$

$$= \lim_{x\to+\infty}\left(\exp\left\{(o(1)-1)\,\alpha^{-1}(x)\right\} - 1\right) \times \exp\left\{\alpha^{-1}(x)\right\} = -\infty,$$

this implies

$$\lim_{r\to+\infty}\exp y = 0.$$

Therefore yielding

$$\lim_{r\to+\infty}\frac{T(r,f_1)}{T(r,f_2)} = 0,$$

that is $T(r,f_1) = o(T(r,f_2))$ as $r\to+\infty$. □

Lemma 3.10. *Let $F(z)\not\equiv 0$, $A_j(z)$ $(j=0,...,k-1)$ be meromorphic functions, and let f be a meromorphic solution of* (2) *satisfying*

$$\max\{\rho_{(\alpha(\log),\beta,\gamma)}[A_j], \rho_{(\alpha(\log),\beta,\gamma)}[F] : j = 0,1,...,k-1\} < \mu_{(\alpha(\log),\beta,\gamma)}[f].$$

Then we have

$$\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f] = \lambda_{(\alpha(\log),\beta,\gamma)}[f] = \mu_{(\alpha(\log),\beta,\gamma)}[f].$$

Proof. By (2), we get that

$$\frac{1}{f} = \frac{1}{F}\left(\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots + A_1(z)\frac{f'}{f} + A_0\right). \quad (13)$$

Now, by (2) it is easy to see that if f has a zero at z_0 of order a $(a>k)$, and if $A_0,...,A_{k-1}$ are analytic at z_0, then $F(z)$ must have a zero at z_0 of order $a-k$, hence

$$n\left(r,\frac{1}{f}\right) \le k\overline{n}\left(r,\frac{1}{f}\right) + n\left(r,\frac{1}{F}\right) + \sum_{j=0}^{k-1} n(r,A_j) \quad (14)$$

and

$$N\left(r,\frac{1}{f}\right) \le k\overline{N}\left(r,\frac{1}{f}\right) + N\left(r,\frac{1}{F}\right) + \sum_{j=0}^{k-1} N(r,A_j). \quad (15)$$

By the lemma on logarithmic derivative ([15], p. 34) and (13), we have

$$m\left(r,\frac{1}{f}\right) \le m\left(r,\frac{1}{F}\right) + \sum_{j=0}^{k-1} m(r,A_j) + O(\log T(r,f) + \log r) \quad (r\notin E_7). \quad (16)$$

where $E_7\subset[0,+\infty)$ is a set of r of finite linear measure. By (15) and (16), we obtain that

$$T(r,f) = T\left(r,\frac{1}{f}\right) + O(1) \le k\overline{N}\left(r,\frac{1}{f}\right) + T(r,F) + \sum_{j=0}^{k-1} T(r,A_j) + O(\log(rT(r,f))) \ (r\notin E_7). \quad (17)$$

Since $\max\{\rho_{(\alpha(\log),\beta,\gamma)}[A_j], \rho_{(\alpha(\log),\beta,\gamma)}[F] : j = 0,1,...,k-1\} < \mu_{(\alpha(\log),\beta,\gamma)}[f]$, then by Lemma 3.9

$$T(r,F) = o(T(r,f)),\ T(r,A_j) = o(T(r,f)) \ (j=0,...,k-1) \text{ as } r\to+\infty. \quad (18)$$

Since f is transcendental, then we have

$$O(\log(rT(r,f))) = o(T(r,f)) \text{ as } r\to+\infty. \quad (19)$$

Therefore, by substituting (18) and (19) into (17), for all $|z| = r\notin E_7$, we get that

$$T(r,f) \le O\left(\overline{N}\left(r,\frac{1}{f}\right)\right).$$

Hence from above we have

$$\mu_{(\alpha(\log),\beta,\gamma)}[f] \le \overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f].$$

Since $\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f] \le \lambda_{(\alpha(\log),\beta,\gamma)}[f] \le \mu_{(\alpha(\log),\beta,\gamma)}[f]$, then

$$\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f] = \lambda_{(\alpha(\log),\beta,\gamma)}[f] = \mu_{(\alpha(\log),\beta,\gamma)}[f].$$

□

Lemma 3.11. *([3]) Let $F(z) \not\equiv 0$, $A_j(z)$ $(j = 0, ..., k-1)$ be entire functions. Also let f be a solution of (2) satisfying $\max\{\rho_{(\alpha(\log),\beta,\gamma)}[A_j], \rho_{(\alpha(\log),\beta,\gamma)}[F] : j = 0, 1, ..., k-1\} < \rho_{(\alpha(\log),\beta,\gamma)}[f]$. Then we have*

$$\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f] = \lambda_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha(\log),\beta,\gamma)}[f].$$

Lemma 3.12. *Let f be a transcendental entire function. Then $\rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha(\log),\beta,\gamma)}[f^{(k)}]$, $k \in \mathbb{N}$.*

Proof. By Lemma 4.4 in ([3]), we have $\rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha(\log),\beta,\gamma)}[f']$, so by using mathematical induction, we easily obtain the result. □

Lemma 3.13. *([3]) Let f be a meromorphic function. If $\rho_{(\alpha,\beta,\gamma)}[f] = \rho < +\infty$, then $\rho_{(\alpha(\log),\beta,\gamma)}[f] = 0$.*

Lemma 3.14. *([17]) Let $A_j(z)$ $(j = 0, ..., k-1)$ be entire coefficients in (1), and at least one of them is transcendental. If $A_s(z)$ $(0 \le s \le k-1)$ is the first one (according to the sequence of $A_0(z), ..., A_{k-1}(z)$) satisfying*

$$\liminf_{r\to+\infty} \frac{\sum_{j=s+1}^{k-1} m(r, A_j)}{m(r, A_s)} < 1, \ r \notin E_8,$$

where E_8 is a set of r of finite linear measure. Then (1) possesses at most s linearly independent entire solutions satisfying

$$\limsup_{r\to+\infty} \frac{\log T(r, f)}{m(r, A_s)} = 0, \ r \notin E_8.$$

4 Proof of the Main Results

Proof of Theorem 2.3. Suppose that f $(\not\equiv 0)$ is a solution of equation (1). By Theorem 2.1, we know that every solution f $(\not\equiv 0)$ of (1) satisfies $\rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha,\beta,\gamma)}[A_0]$. So, we only need to prove that every solution f $(\not\equiv 0)$ of (1) satisfies $\mu_{(\alpha(\log),\beta,\gamma)}[f] = \mu_{(\alpha,\beta,\gamma)}[A_0]$. First, we prove that $\mu_1 = \mu_{(\alpha(\log),\beta,\gamma)}[f] \ge \mu_{(\alpha,\beta,\gamma)}[A_0] = \mu_0$. Suppose the contrary. Set $\max\{\rho_{(\alpha,\beta,\gamma)}[A_j] : j = 1, ..., k-1, \mu_{(\alpha(\log),\beta,\gamma)}[f]\} = \rho < \mu_{(\alpha,\beta,\gamma)}[A_0] = \mu_0$. From (1), we can write

$$|A_0(z)| \le \left|\frac{f^{(k)}}{f}\right| + |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right|. \tag{20}$$

For any given $\varepsilon\,(0 < 2\varepsilon < \mu_0 - \rho)$ and for sufficiently large r, we have

$$|A_0(z)| > \exp^{[2]}\left\{\alpha^{-1}\left((\mu_0 - \varepsilon)\,\beta(\log\gamma(r))\right)\right\} \tag{21}$$

and

$$|A_j(z)| \le \exp^{[2]}\left\{\alpha^{-1}\left(\left(\rho + \frac{\varepsilon}{2}\right)\beta(\log\gamma(r))\right)\right\}, \quad j \in \{1, 2, ..., k-1\}. \tag{22}$$

By Lemma 3.1, there exist a constant $B > 0$ and a set $E_1 \subset (1, +\infty)$ having finite logarithmic measure such that for all z satisfying $|z| = r \notin [0, 1] \cup E_1$, we have

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le B\,[T(2r, f)]^{k+1} \quad (j = 1, 2, ..., k). \tag{23}$$

It follows by Lemma 3.2 and (23), that for sufficiently large $|z| = r \in E_2 \setminus (E_1 \cup [0, 1])$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \le B\,[T(2r, f)]^{k+1}$$
$$\le B\left[\exp^{[2]}\left\{\alpha^{-1}\left(\left(\mu_1 + \frac{\varepsilon}{2}\right)\beta(\log\gamma(r))\right)\right\}\right]^{k+1} \quad (j = 1, 2, ..., k), \tag{24}$$

where E_2 is a set of infinite logarithmic measure. Hence, by substituting (21)-(24) into (20), for the above $\varepsilon\,(0 < 2\varepsilon < \mu_0 - \rho)$, we obtain for sufficiently large $|z| = r \in E_2 \setminus (E_1 \cup [0, 1])$

$$\exp^{[2]}\left\{\alpha^{-1}\left((\mu_0 - \varepsilon)\,\beta(\log\gamma(r))\right)\right\}$$
$$\le Bk\exp^{[2]}\left\{\alpha^{-1}\left(\left(\rho + \frac{\varepsilon}{2}\right)\beta(\log\gamma(r))\right)\right\} \times [T(2r, f)]^{k+1}$$
$$\le Bk\exp^{[2]}\left\{\alpha^{-1}\left(\left(\rho + \frac{\varepsilon}{2}\right)\beta(\log\gamma(r))\right)\right\} \times \left[\exp^{[2]}\left\{\alpha^{-1}\left(\left(\mu_1 + \frac{\varepsilon}{2}\right)\beta(\log\gamma(r))\right)\right\}\right]^{k+1}$$
$$\le \exp^{[2]}\left\{\alpha^{-1}\left((\rho + \varepsilon)\,\beta(\log\gamma(r))\right)\right\}. \tag{25}$$

Since $E_2 \setminus (E_1 \cup [0, 1])$ is a set of infinite logarithmic measure, then there exists a sequence of points $|z_n| = r_n \in E_2 \setminus (E_1 \cup [0, 1])$ tending to $+\infty$. It follows by (25) that

$$\exp^{[2]}\left\{\alpha^{-1}\left((\mu_0 - \varepsilon)\,\beta(\log\gamma(r_n))\right)\right\} \le \exp^{[2]}\left\{\alpha^{-1}\left((\rho + \varepsilon)\,\beta(\log\gamma(r_n))\right)\right\} \tag{26}$$

holds for all z_n satisfying $|z_n| = r_n \in E_2 \setminus (E_1 \cup [0, 1])$ as $|z_n| \to +\infty$. By arbitrariness of $\varepsilon > 0$ and the monotony of the function α^{-1}, from (26) we obtain that $\rho \ge \mu_{(\alpha,\beta,\gamma)}[A_0] = \mu_0$. This contradiction proves the inequality $\mu_{(\alpha(\log),\beta,\gamma)}[f] \ge \mu_{(\alpha,\beta,\gamma)}[A_0]$.

Now, we prove $\mu_{(\alpha(\log),\beta,\gamma)}[f] \leq \mu_{(\alpha,\beta,\gamma)}[A_0] = \mu_0$. By (1), we have

$$\left|\frac{f^{(k)}}{f}\right| \leq |A_{k-1}(z)| \left|\frac{f^{(k-1)}}{f}\right| + \cdots + |A_1(z)| \left|\frac{f'}{f}\right| + |A_0(z)| . \tag{27}$$

By Lemma 3.5, there exists a set $E_4 \subset (1,+\infty)$ of finite logarithmic measure such that the estimation

$$\frac{f^{(j)}(z)}{f(z)} = \left(\frac{\nu_f(r)}{z}\right)^j (1+o(1)) \quad (j=1,...,k) \tag{28}$$

holds for all z satisfying $|z| = r \notin E_4$, $r \to +\infty$ and $|f(z)| = M(r,f)$. By Lemma 3.3, for any given $\varepsilon > 0$, there exists a set $E_3 \subset (1,+\infty)$ that has infinite logarithmic measure, such that

$$|A_0(z)| \leq \exp^{[2]}\left\{\alpha^{-1}\left(\left(\mu_0 + \frac{\varepsilon}{2}\right)\beta(\log\gamma(r))\right)\right\} \tag{29}$$

and for sufficiently large r

$$\begin{aligned}|A_j(z)| &\leq \exp^{[2]}\left\{\alpha^{-1}\left(\left(\rho + \frac{\varepsilon}{2}\right)\beta(\log\gamma(r))\right)\right\} \\ &\leq \exp^{[2]}\left\{\alpha^{-1}\left(\left(\mu_0 + \frac{\varepsilon}{2}\right)\beta(\log\gamma(r))\right)\right\} \\ &\quad (j=1,...,k-1).\end{aligned} \tag{30}$$

Substituting (28), (29) and (30) into (27), we obtain

$$\begin{aligned}\nu_f(r) &\leq kr^k |1+o(1)| \exp^{[2]}\left\{\alpha^{-1}\left(\left(\mu_0 + \frac{\varepsilon}{2}\right.\right.\right. \\ &\qquad \left.\left.\left. \times\beta(\log\gamma(r))\right)\right\} \\ &\leq \exp^{[2]}\left\{\alpha^{-1}\left((\mu_0+\varepsilon)\beta(\log\gamma(r))\right)\right\}\end{aligned} \tag{31}$$

for all z satisfying $|z| = r \in E_3 \backslash E_4$, $r \to +\infty$ and $|f(z)| = M(r,f)$. By Lemma 3.4, from (31) we obtain for each $\varepsilon > 0$

$$\begin{aligned}T(r,f) &\leq \log M(r,f) < \log[\mu(r)(\nu_f(2r)+2)] \\ &= \log\left[\left|a_{\nu_f(r)}\right| r^{\nu_f(r)}(\nu_f(2r)+2)\right] \\ &< \nu_f(r)\log r + \log(2\nu_f(2r)) + \log\left|a_{\nu_f(r)}\right| \\ &\leq \exp^{[2]}\left\{\alpha^{-1}\left((\mu_0+\varepsilon)\beta(\log\gamma(r))\right)\right\}\log r \\ &\quad + \log\left(2\exp^{[2]}\left\{\alpha^{-1}\left((\mu_0+\varepsilon)\beta(\log\gamma(2r))\right)\right\}\right) \\ &\quad + \log\left|a_{\nu_f(r)}\right| \\ &\leq \exp^{[2]}\left\{\alpha^{-1}\left((\mu_0+2\varepsilon)\beta(\log\gamma(r))\right)\right\} + \log 2 \\ &\quad + \exp\left\{\alpha^{-1}\left((\mu_0+\varepsilon)\beta(\log\gamma(2r))\right)\right\} \\ &\quad + \log\left|a_{\nu_f(r)}\right| \\ &\leq \exp^{[2]}\left\{\alpha^{-1}\left((\mu_0+3\varepsilon)\beta(\log\gamma(r))\right)\right\}.\end{aligned}$$

Hence,

$$\frac{\alpha(\log^{[2]} T(r,f))}{\beta(\log\gamma(r))} \leq \mu_0 + 3\varepsilon.$$

It follows

$$\mu_{(\alpha(\log),\beta,\gamma)}[f] = \liminf_{r\longrightarrow+\infty} \frac{\alpha(\log^{[2]} T(r,f))}{\beta(\log\gamma(r))} \leq \mu_0 + 3\varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then we obtain $\mu_{(\alpha(\log),\beta,\gamma)}[f] \leq \mu_0$. Hence every solution $f \not\equiv 0$ of equation (1) satisfies $\mu_{(\alpha,\beta,\gamma)}[A_0] = \mu_{(\alpha(\log),\beta,\gamma)}[f] \leq \rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha,\beta,\gamma)}[A_0]$.

Secondly, we prove that $\underline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f-g] = \mu_{(\alpha(\log),\beta,\gamma)}[f]$ and

$$\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f-g] = \rho_{(\alpha(\log),\beta,\gamma)}[f].$$

Set $h = f - g$. Since

$$\begin{aligned}\rho_{(\alpha(\log),\beta,\gamma)}[g] < \mu_{(\alpha,\beta,\gamma)}[A_0] &= \mu_{(\alpha(\log),\beta,\gamma)}[f] \\ &\leq \rho_{(\alpha(\log),\beta,\gamma)}[f],\end{aligned}$$

it follows from Proposition 1.8 and Proposition 1.10 that $\rho_{(\alpha(\log),\beta,\gamma)}[h] = \rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha,\beta,\gamma)}[A_0]$ and $\mu_{(\alpha(\log),\beta,\gamma)}[h] = \mu_{(\alpha(\log),\beta,\gamma)}[f] = \mu_{(\alpha,\beta,\gamma)}[A_0]$. By substituting $f = g + h$, $f' = g' + h'$, $\ldots$, $f^{(k)} = g^{(k)} + h^{(k)}$ into (1), we obtain

$$\begin{aligned}&h^{(k)} + A_{k-1}(z)h^{(k-1)} + \cdots + A_0(z)h \\ &= -\left(g^{(k)} + A_{k-1}(z)g^{(k-1)} + \cdots + A_0(z)g\right).\end{aligned} \tag{32}$$

If $g^{(k)} + A_{k-1}(z)g^{(k-1)} + \cdots + A_0(z)g = G \equiv 0$, then by the first part of the proof of Theorem 2.3 we have $\rho_{(\alpha(\log),\beta,\gamma)}[g] \geq \mu_{(\alpha,\beta,\gamma)}[A_0]$ which contradicts the assumption $\rho_{(\alpha(\log),\beta,\gamma)}[g] < \mu_{(\alpha,\beta,\gamma)}[A_0]$. Hence $G \not\equiv 0$. By Proposition 1.8, Lemma 3.12 and Lemma 3.13, we get

$$\begin{aligned}&\rho_{(\alpha(\log),\beta,\gamma)}[G] \\ &\leq \max\{\rho_{(\alpha(\log),\beta,\gamma)}[g], \rho_{(\alpha(\log),\beta,\gamma)}(A_j) \\ &\quad (j=0,1,...,k-1)\} \\ &= \rho_{(\alpha(\log),\beta,\gamma)}[g] < \mu_{(\alpha,\beta,\gamma)}[A_0] = \mu_{(\alpha(\log),\beta,\gamma)}[f] \\ &= \mu_{(\alpha(\log),\beta,\gamma)}[h] \leq \rho_{(\alpha(\log),\beta,\gamma)}[h] = \rho_{(\alpha(\log),\beta,\gamma)}[f] \\ &= \rho_{(\alpha,\beta,\gamma)}[A_0].\end{aligned}$$

Then, it follows from Lemma 3.10, Lemma 3.11 and (32) that $\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[h] = \lambda_{(\alpha(\log),\beta,\gamma)}[h] = \rho_{(\alpha(\log),\beta,\gamma)}(h) = \rho_{(\alpha(\log),\beta,\gamma)}[f]$ and

$$\begin{aligned}&\overline{\underline{\lambda}}_{(\alpha(\log),\beta,\gamma)}[h] = \underline{\lambda}_{(\alpha(\log),\beta,\gamma)}[h] = \mu_{(\alpha(\log),\beta,\gamma)}[h] \\ &= \mu_{(\alpha(\log),\beta,\gamma)}[f].\end{aligned}$$

Therefore, $\underline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f-g] = \mu_{(\alpha(\log),\beta,\gamma)}[f]$ and

$$\overline{\lambda}_{(\alpha(\log),\beta,\gamma)}[f-g] = \rho_{(\alpha(\log),\beta,\gamma)}[f]$$

which completes the proof of Theorem 2.3.

Proof of Theorem 2.4. Suppose that $f\ (\not\equiv 0)$ is a solution of equation (1). Then by Theorem 2.2, we obtain $\rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha,\beta,\gamma)}[A_0]$. Now, we prove that $\mu_1 = \mu_{(\alpha(\log),\beta,\gamma)}[f] \geq \mu_{(\alpha,\beta,\gamma)}[A_0] = \mu_0$. Suppose the contrary $\mu_1 = \mu_{(\alpha(\log),\beta,\gamma)}[f] < \mu_{(\alpha,\beta,\gamma)}[A_0] = \mu_0$. We set $b = \max\{\rho_{(\alpha,\beta,\gamma)}[A_j] : \rho_{(\alpha,\beta,\gamma)}[A_j] < \mu_{(\alpha,\beta,\gamma)}[A_0]\}$. If $\rho_{(\alpha,\beta,\gamma)}[A_j] < \mu_{(\alpha,\beta,\gamma)}[A_0]$, then for any given ε with $0 < 3\varepsilon < \min\{\mu_0 - b, \tau - \tau_1\}$ and for sufficiently large r, we have

$$|A_j(z)| \leq \exp^{[2]}\left\{\alpha^{-1}\left((b+\varepsilon)\beta(\log\gamma(r))\right)\right\}$$
$$\leq \exp^{[2]}\left\{\alpha^{-1}\left(\left(\mu_{(\alpha,\beta,\gamma)}[A_0]-2\varepsilon\right)\beta(\log\gamma(r))\right)\right\}. \tag{33}$$

If $\rho_{(\alpha,\beta,\gamma)}[A_j] = \mu_{(\alpha,\beta,\gamma)}[A_0]$, $\tau_{(\alpha,\beta,\gamma),M}[A_j] \leq \tau_1 < \underline{\tau}_{(\alpha,\beta,\gamma),M}[A_0] = \tau$, then for sufficiently large r, we have

$$|A_j(z)| \leq \exp^{[2]}\left\{\alpha^{-1}\left(\log\left((\tau_1+\varepsilon)\right.\right.\right.$$
$$\left.\left.\left.\times\ \left(\exp\{\beta(\log\gamma(r))\}\right)^{\mu_0}\right)\right)\right\} \tag{34}$$

and

$$|A_0(z)| > \exp^{[2]}\left\{\alpha^{-1}\left(\log\left((\tau-\varepsilon)\right.\right.\right.$$
$$\left.\left.\left.\times\ \left(\exp\{\beta(\log\gamma(r))\}\right)^{\mu_0}\right)\right)\right\}. \tag{35}$$

By Lemma 3.1 and Lemma 3.2, for any given ε with $0 < \varepsilon < \mu_0 - \mu_1$ and sufficiently large $|z| = r \in E_2\setminus(E_1\cup[0,1])$

$$\left|\frac{f^{(j)}(z)}{f(z)}\right| \leq B\,[T(2r,f)]^{k+1}$$
$$\leq B\left[\exp^{[2]}\left\{\alpha^{-1}\left((\mu_1+\varepsilon)\beta(\log\gamma(r))\right)\right\}\right]^{k+1}$$
$$(j=1,2,...,k), \tag{36}$$

where E_2 is a set of infinite logarithmic measure. Hence, by substituting (33)-(36) into (20), for the above ε with $0 < \varepsilon < \min\left\{\frac{\mu_0-b}{3}, \frac{\tau-\tau_1}{3}, \mu_0-\mu_1\right\}$, we obtain for sufficiently large $|z| = r \in E_2\setminus(E_1\cup[0,1])$

$$\exp^{[2]}\left\{\alpha^{-1}\left(\log\left((\tau-\varepsilon)\left(\exp\{\beta(\log\gamma(r))\}\right)^{\mu_0}\right)\right)\right\}$$
$$\leq Bk\exp^{[2]}\left\{\alpha^{-1}\left(\log\left((\tau_1+\varepsilon)\right.\right.\right.$$
$$\left.\left.\left.\times\left(\exp\{\beta(\log\gamma(r))\}\right)^{\mu_0}\right)\right)\right\}[T(2r,f)]^{k+1}$$
$$\leq Bk\exp^{[2]}\left\{\alpha^{-1}\left(\log\left((\tau_1+\varepsilon)\right.\right.\right.$$
$$\left.\left.\left.\times\left(\exp\{\beta(\log\gamma(r))\}\right)^{\mu_0}\right)\right)\right\}$$
$$\times\left[\exp^{[2]}\left\{\alpha^{-1}\left((\mu_1+\varepsilon)\beta(\log\gamma(r))\right)\right\}\right]^{k+1}$$
$$\leq \exp^{[2]}\left\{\alpha^{-1}\left(\log\left((\tau_1+2\varepsilon)\right.\right.\right.$$
$$\left.\left.\left.\times\left(\exp\{\beta(\log\gamma(r))\}\right)^{\mu_0}\right)\right)\right\}. \tag{37}$$

Since $E_2\setminus(E_1\cup[0,1])$ is a set of infinite logarithmic measure, then there exists a sequence of points $|z_n| = r_n \in E_2\setminus(E_1\cup[0,1])$ tending to $+\infty$. It follows by (37) that

$$\exp^{[2]}\left\{\alpha^{-1}\left(\log\left((\tau-\varepsilon)\left(\exp\{\beta(\log\gamma(r_n))\}\right)^{\mu_0}\right)\right)\right\}$$
$$\leq \exp^{[2]}\left\{\alpha^{-1}\left(\log\left((\tau_1+2\varepsilon)\right.\right.\right.$$
$$\left.\left.\left.\times\left(\exp\{\beta(\log\gamma(r_n))\}\right)^{\mu_0}\right)\right)\right\}$$

holds for all z_n satisfying $|z_n| = r_n \in E_2\setminus(E_1\cup[0,1])$ as $|z_n|\to+\infty$. By arbitrariness of $\varepsilon > 0$ and the monotonicity of the function α^{-1}, we obtain that $\tau_1 \geq \tau$. This contradiction proves the inequality $\mu_{(\alpha(\log),\beta,\gamma)}[f] \geq \mu_{(\alpha,\beta,\gamma)}[A_0]$.
Now, we prove $\mu_{(\alpha(\log),\beta,\gamma)}[f] \leq \mu_{(\alpha,\beta,\gamma)}[A_0]$. By using similar arguments as in the proofs of Theorem 2.3, we obtain $\mu_{(\alpha(\log),\beta,\gamma)}[f] \leq \mu_{(\alpha,\beta,\gamma)}[A_0]$. Hence, every solution $f\not\equiv 0$ of equation (1) satisfies

$$\mu_{(\alpha,\beta,\gamma)}[A_0] = \mu_{(\alpha(\log),\beta,\gamma)}[f] \leq \rho_{(\alpha(\log),\beta,\gamma)}[f]$$
$$= \rho_{(\alpha,\beta,\gamma)}[A_0].$$

The second part of the proof of Theorem 2.3 completes the proof of Theorem 2.4.

Proof of Theorem 2.5. Suppose that $f\ (\not\equiv 0)$ is a solution of equation (1). We divide the proof into two parts: (i) $\rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha,\beta,\gamma)}[A_0]$, (ii) $\mu_{(\alpha(\log),\beta,\gamma)}[f] = \mu_{(\alpha,\beta,\gamma)}[A_0]$.
(i) First, we prove that $\rho_1 = \rho_{(\alpha(\log),\beta,\gamma)}[f] \geq \rho_{(\alpha,\beta,\gamma)}[A_0] = \rho_0$. Suppose the contrary $\rho_1 = \rho_{(\alpha(\log),\beta,\gamma)}[f] < \rho_{(\alpha,\beta,\gamma)}[A_0] = \rho_0$. From (1), we can write

$$A_0(z) = -\left(\frac{f^{(k)}}{f} + A_{k-1}(z)\frac{f^{(k-1)}}{f} + \cdots\right.$$
$$\left. + A_1(z)\frac{f'}{f}\right). \tag{38}$$

By Lemma 3.6 and (38), we have

$$m(r,A_0) \leq \sum_{j=1}^{k-1} m(r,A_j) + \sum_{j=1}^{k} m\left(r,\frac{f^{(j)}}{f}\right) + \log k$$

$$\leq \sum_{j=1}^{k-1} m(r, A_j) + O\left(\exp\left\{\alpha^{-1}\left(\left(\rho_1 + \frac{\varepsilon}{2}\right)\right.\right.\right.$$
$$\left.\left.\left.\times \beta(\log\gamma(r))\right)\right\}\right) \quad (39)$$

holds possibly outside of an exceptional set $E_5 \subset (0, +\infty)$ with finite linear measure. Suppose that

$$\limsup_{r\to+\infty} \frac{\sum_{j=1}^{k-1} m(r, A_j)}{m(r, A_0)} = \sigma < \kappa < 1.$$

Then for sufficiently large r, we have

$$\sum_{j=1}^{k-1} m(r, A_j) < \kappa m(r, A_0). \quad (40)$$

By (39) and (40), we have

$$(1-\kappa)\, m(r, A_0)$$
$$\leq O\left(\exp\left\{\alpha^{-1}\left(\left(\rho_1 + \frac{\varepsilon}{2}\right)\beta(\log\gamma(r))\right)\right\}\right),$$
$$r \notin E_5.$$

It follows that

$$T(r, A_0) = m(r, A_0)$$
$$\leq \exp\left\{\alpha^{-1}\left((\rho_1+\varepsilon)\beta(\log\gamma(r))\right)\right\},\ r \notin E_5. \quad (41)$$

Hence

$$\frac{\alpha(\log T(r, A_0))}{\beta(\log\gamma(r))} \leq \rho_1 + \varepsilon$$

and

$$\rho_{(\alpha,\beta,\gamma)}[A_0] = \limsup_{r\longrightarrow+\infty} \frac{\alpha(\log T(r, A_0))}{\beta(\log\gamma(r))} \leq \rho_1 + \varepsilon.$$

Since $\varepsilon > 0$ is arbitrary, then we obtain $\rho_{(\alpha,\beta,\gamma)}[A_0] \leq \rho_1$. This contradiction proves the inequality $\rho_{(\alpha(\log),\beta,\gamma)}[f] \geq \rho_{(\alpha,\beta,\gamma)}[A_0]$. On the other hand, by Lemma 3.7, we have

$$\rho_{(\alpha(\log),\beta,\gamma)}[f]$$
$$\leq \max\{\rho_{(\alpha,\beta,\gamma)}[A_j] : j = 0, 1, ..., k-1\} \quad (42)$$
$$= \rho_{(\alpha,\beta,\gamma)}[A_0].$$

Hence every solution $f \not\equiv 0$ of equation (1) satisfies $\rho_{(\alpha(\log),\beta,\gamma)}[f] = \rho_{(\alpha,\beta,\gamma)}[A_0]$.
(ii) By using similar arguments as in the proofs of Theorem 2.3, we obtain $\mu_{(\alpha(\log),\beta,\gamma)}[f] = \mu_{(\alpha,\beta,\gamma)}[A_0]$. Hence, every solution $f \not\equiv 0$ of equation (1) satisfies

$$\mu_{(\alpha,\beta,\gamma)}[A_0] = \mu_{(\alpha(\log),\beta,\gamma)}[f] \leq \rho_{(\alpha(\log),\beta,\gamma)}[f]$$
$$= \rho_{(\alpha,\beta,\gamma)}[A_0].$$

The second part of the proof of Theorem 2.3 completes the proof of Theorem 2.5.

Proof of Theorem 2.6. By Lemma 3.14, we obtain that every linearly independent solution of (1) satisfies $\limsup_{r\to+\infty} \frac{\log T(r,f)}{m(r,A_0)} > 0,\ r \notin E_8$. So, every solution $f\ (\not\equiv 0)$ of (1) satisfies $\limsup_{r\to+\infty} \frac{\log T(r,f)}{m(r,A_0)} > 0$, $r \notin E_8$. Hence, there exist $\delta > 0$ and a sequence $\{r_n\}_{n=1}^{+\infty}$ tending to ∞ such that for sufficiently large $r_n \notin E_8$ and for every solution $f\ (\not\equiv 0)$ of (1), we have

$$\log T(r_n, f) > \delta m(r_n, A_0). \quad (43)$$

Since $\mu_{(\alpha,\beta,\gamma)}[A_0] = \rho_{(\alpha,\beta,\gamma)}[A_0]$, then by (43), for any given $\varepsilon > 0$ and sufficiently large $r_n \notin E_8$, we get

$$\log T(r_n, f)$$
$$> \delta \exp\left\{\alpha^{-1}\left(\left(\mu_{(\alpha,\beta,\gamma)}[A_0] - \frac{\varepsilon}{2}\right)\beta(\log\gamma(r_n))\right)\right\}$$
$$\geq \exp\left\{\alpha^{-1}\left((\mu_{(\alpha,\beta,\gamma)}[A_0] - \varepsilon)\beta(\log\gamma(r_n))\right)\right\},$$

which implies

$$\rho_{(\alpha(\log),\beta,\gamma)}[f] \geq \mu_{(\alpha,\beta,\gamma)}[A_0] = \rho_{(\alpha,\beta,\gamma)}[A_0]. \quad (44)$$

On the other hand, by Lemma 3.7, we have

$$\rho_{(\alpha(\log),\beta,\gamma)}[f]$$
$$\leq \max\{\rho_{(\alpha,\beta,\gamma)}[A_j] : j = 0, 1, ..., k-1\} \quad (45)$$
$$= \mu_{(\alpha,\beta,\gamma)}[A_0] = \rho_{(\alpha,\beta,\gamma)}[A_0].$$

By (44) and (45), we obtain $\rho_{(\alpha(\log),\beta,\gamma)}[f] = \mu_{(\alpha,\beta,\gamma)}[A_0] = \rho_{(\alpha,\beta,\gamma)}[A_0]$.

The second part of the proof of Theorem 2.3 completes the proof of Theorem 2.6.

5 Conclusion

Throughout this article, by using the concepts of lower (α, β, γ)-order and lower (α, β, γ)-type, we obtain some growth and oscillation properties of solutions of higher order linear differential equations in which the coefficients are entire functions. We improve and extend some recently obtained results by the author and Biswas [4]. Inspired by the results already established, one may explore analogous theorems for differential equations in which the coefficients are meromorphic functions of (α, β, γ)-order. Further, we can study differential polynomials generated by solutions of the differential equations (1) and (2) when the coefficients of these equations are entire, meromorphic or analytic functions in the unit disc.

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