

A note on basis problem in normed spaces

VINICIUS COELHO¹, JOILSON RIBEIRO², LUCIANA SALGADO³

¹Centro Multidisciplinar de Bom Jesus da Lapa
Universidade Federal do Oeste da Bahia
Av. Manoel Novais, 1064, Centro
47600-000 - Bom Jesus da Lapa-BA, BRAZIL

²Instituto de Matemática e Estatística
Universidade Federal da Bahia
Av. Adhemar de Barros, S/N, Ondina
40170-110 - Salvador-BA, BRAZIL

³Instituto de Matemática e Estatística
Universidade Federal do Rio de Janeiro
Avenida Athos da Silveira Ramos 149 Cidade Universitária
P.O. Box 68530, 21941-909 Rio de Janeiro-RJ, BRAZIL

Abstract: In this note, we extended the well known criterion of Banach-Grubblum and the Bessaga-Pełczyński Theorem to normed spaces context, not necessarily complete (Banach) one. As application, we show the Principle of Selection of Bessaga-Pełczyński for normed spaces and the Spectral Theorem for compact self-adjoint operators on inner product spaces.

Key-Words: Normed spaces, Schauder basis, Spectral theorem

Received: March 11, 2024. Revised: September 13, 2024. Accepted: October 12, 2024. Published: November 22, 2024.

1 Introduction

In 1933, S. Banach affirmed that *every infinitely dimensional Banach space contains an infinite dimensional subspace with a basis*, without proof. In 1958, Bessaga and Pełczyński [3] developed several generalizations and modifications of Banach's claim, and proved their well known Selection Principle in this setting.

In this paper, as a short complement of this theory, we work on Banach's problem for normed (not necessarily complete) spaces, giving an expected generalization of the previous results, but not formally proved yet, as long as we know. We also generalize Banach-Grubblum's criterion in this context, and give a Spectral Theorem on inner product spaces.

Vector space basis play a key role in the most varied problems of functional analysis, as example we refer the reader to see [3, 12, 16, 17, 18, 19].

For any vector space, it is well known that there exists an algebraic basis (or Hamel basis). However, there is another notion of basis, due to J. Schauder [16, 17], defined as follows:

Definition 1. A sequence $(x_n)_{n=1}^{\infty}$ in a normed space X is called a *Schauder basis* for X if for each x in X there is a unique sequence (a_n) of scalars such that $x = \sum_{i=1}^{\infty} a_n x_n$.

The uniqueness of the representation allows us to consider the linear operator for each n in \mathbb{N} :

$$x_n^* : X \rightarrow \mathbb{K}, x_n^* \left(\sum_{i=1}^{\infty} a_j x_j \right) = a_n,$$

this operators are called *coefficient operators* (or *coordinates operator*).

Let $(x_n)_{n=1}^{\infty}$ be a Schauder basis in the normed space $(X, \|\cdot\|)$, and consider the

linear space $\mathcal{L}_X = \{(a_n)_{n=1}^\infty \mid \sum_{i=1}^\infty a_i x_i$ is convergent}. A computation shows that the function $\eta : \mathcal{L}_X \rightarrow \mathbb{R}$ given by $\eta((a_n)_{n=1}^\infty) := \sup \left\{ \left\| \sum_{i=1}^n a_i x_i \right\| : n \in \mathbb{N} \right\}$ gives a norm in \mathcal{L}_X .

In this paper, our purpose is to provide the criterion of Banach-Grunblum and the Bessaga-Pełczyński Theorem for normed spaces, generalizing the Banach's version [2, ?]. In the conclusion, we explain possible applications of this theory to pure and applied mathematical sciences, as the combination of least-squares method and a Schauder basis to provide a numerical solution for a wide class of linear differential or integral equations (see [15]).

To deduce these results, we start with the following broad notion of Schauder basis.

Definition 2. Let X be a normed space and $(x_n)_{n=1}^\infty$ be a Schauder basis in X . We say that $(x_i)_{i=1}^\infty$ is an *essential Schauder basis* for X if $T_X : \mathcal{L}_X \rightarrow X$ given by $T_X((a_n)_{n=1}^\infty) = \sum_{n=1}^\infty a_n x_n$ is an isomorphism.

It is not difficult to show that in Banach spaces, every Schauder basis is an essential Schauder basis. This identification is important because, with it, any space that admits a Schauder basis can be seen as a space of sequences. In the original definition of Schauder, there was the requirement that coordinates functional should be continuous. However, in 1932 Banach [2, pag 111] showed that in complete normed spaces this is always true. But, if the space is not complete, this assertion is false (see [5, Example 12.5]).

A sequence $(x_n)_{n=1}^\infty$ may not be a Schauder basis for a normed space X , because $\overline{[x_n : n \in \mathbb{N}]}$ does not reach all the space X . In this case, we say that:

Definition 3. A sequence $(x_n)_{n=1}^\infty$ in a normed space X is a *basic sequence* if the sequence $(x_n)_n$ is a Schauder basis for $\overline{[x_n : n \in \mathbb{N}]}$.

In Banach spaces theory, we have the following practical and useful criterion for de-

termining whether a given sequence is basic or not.

Theorem 1.1. (*Banach-Grunblum's Criterion*) A sequence $(x_n)_{n=1}^\infty$ of non null vector in a Banach space X is a basic sequence if, and only if, there exists $M \geq 1$ such that for all sequence of scalar $(a_n)_{n=1}^\infty$:

$$\left\| \sum_{i=1}^m a_i x_i \right\| \leq M \left\| \sum_{i=1}^n a_i x_i \right\| \quad (1)$$

whenever $n \geq m$.

Proof. See e.g. [1, Proposition 1.1.9] or [4, Theorem 10.3.13]. \square

One of the consequences of the above theorem was proved by S. Banach in 1932 and another one, by M. Grunblum in 1941. We generalize the Theorem 1.1 for normed spaces setting. For this purpose, it is necessary to introduce the concept of essential basic sequence.

Definition 4. A sequence $(x_n)_{n=1}^\infty$ in a normed space X is called an *essential basic sequence* if it is an essential Schauder basis for $\overline{[x_n : n \in \mathbb{N}]}$.

Note that if X is a Banach space, then every basic sequence is an essential basic sequence.

Let $(S, \|\cdot\|)$ be a normed space. We denote by \widehat{S} the completion of S such that S is dense in \widehat{S} .

Theorem A. (*Banach-Grunblum's criterion for normed spaces*) Let $(x_n)_{n=1}^\infty$ be a sequence of non null vectors in a normed space X . Then the following conditions are equivalents.

- (i) $(x_n)_{n=1}^\infty$ is an essential Schauder basis for $\overline{[x_n : n \in \mathbb{N}]} \subseteq \widehat{X}$;
- (ii) $(x_n)_{n=1}^\infty$ is an essential basic sequence in X ;
- (iii) there exists $M \geq 1$ such that for all sequence of scalar $(a_n)_{n=1}^\infty$:

$$\left\| \sum_{i=1}^m a_i x_i \right\| \leq M \left\| \sum_{i=1}^n a_i x_i \right\| \quad (2)$$

whenever $n \geq m$.

As it has been said before, in his classic book [2], Banach announced, without proof, that in every complete normed (Banach) space of infinite dimension there is an infinite dimensional subspace with Schauder basis. The proof of this result only appeared in the literature in 1958, in a celebrated article by Bessaga and Pełczyński [3] (in the same year Bernard R. Gelbaum [10] also presented another proof). The demonstration presented in [3] is a consequence of the main result of their work, which became known as *selection's principle of Bessaga-Pełczyński*.

Here, we show the Bessaga-Pełczyński's Theorem for normed spaces, and so the Selection Principle for normed spaces, as an application.

We introduce the definition of *equivalent sequence* for Essential Schauder's Basis.

Definition 5. Let $(x_n)_{n=1}^\infty$ be an essential Schauder's basis in a normed space X and $(y_n)_{n=1}^\infty$ be an essential Schauder's basis in a normed space Y . We say that $(x_n)_{n=1}^\infty$ is *equivalent* to $(y_n)_{n=1}^\infty$. In this case, we write $(x_n)_{n=1}^\infty \approx (y_n)_{n=1}^\infty$, to means that, for any scalar's sequence, the series $\sum_{n=1}^\infty a_n x_n$ is convergent in $\overline{\{x_n : n \in \mathbb{N}\}} \subseteq X$ if, and only if, the series $\sum_{n=1}^\infty a_n y_n$ is convergent in $\overline{\{y_n : n \in \mathbb{N}\}} \subseteq Y$.

We are able to present the Bessaga-Pełczyński Theorem for normed spaces.

Theorem B. *Let X be a normed space, $(x_n)_{n=1}^\infty$ be an essential basic sequence in X , and $(x_n^*)_{n=1}^\infty$ be the functional coefficients. If $(y_n)_{n=1}^\infty$ is a sequence in X such that*

$$0 < \sum_{n=1}^\infty \|x_n - y_n\| \cdot \|x_n^*\| =: \lambda < 1 \quad (3)$$

then $(y_n)_{n=1}^\infty$ is an essential basic sequence in X equivalent to $(x_n)_{n=1}^\infty$.

2 Applications

2.1 Bessaga-Pełczyński's Selection Principle

Using Theorem A and B, we obtain the following Bessaga-Pełczyński's Selection Principle and its consequence for normed spaces, as follow.

Definition 6. Let $(x_n)_{n=1}^\infty$ be an essential Schauder's basis in a normed space X , and $(k_n)_{n=0}^\infty$ be a sequence strictly increasing of positive integers, with $k_0 = 0$. A sequence of non null vectors $(y_n)_{n=1}^\infty$ in X is called *essential block basic sequence relative to $(x_n)_{n=1}^\infty$* if

$$y_n = \sum_{i=k_{n-1}+1}^{k_n} b_i x_i$$

where $b_i \in \mathbb{K}$.

The Bessaga-Pełczyński's Selection Principle is the following.

Theorem 2.1. *Let $(x_n)_{n=1}^\infty$ be an essential Schauder basis in a normed space X and $(x_n^*)_{n=1}^\infty$ be the functional coefficients. If $(y_n)_{n=1}^\infty$ is a sequence in X such that $\inf_n \|y_n\| > 0$ and*

$$\lim_{n \rightarrow \infty} x_i^*(y_n) = 0 \text{ for all } i \in \mathbb{N}$$

then $(y_n)_{n=1}^\infty$ contains an essential basic subsequence equivalent to block basic sequence relative to $(x_n)_{n=1}^\infty$.

Corollary 2.2. *Let X be a normed space, $(y_n)_{n=1}^\infty$ is a sequence in X such that $\inf_n \|y_n\| > 0$ and $y_n \rightarrow 0$ weakly. Then $(y_n)_{n=1}^\infty$ contains an essential basic subsequence.*

2.2 Banach problem for normed spaces

As mentioned before, the Bessaga and Pełczyński's proof that in every Banach space of infinite dimension there is an infinite-dimensional subspace with Schauder basis is a consequence of the nowadays known as the Bessaga-Pełczyński selection principle. As we have obtained this result for normed spaces, it is possible to follow the same steps and to show the same result in this broad setting. We stress that this is already known result (see [6]). But here, the proof is done in an elementary way, by using only the Banach-Grünblum's Criterion for normed spaces (Theorem A).

Theorem 2.3. *Every normed space contains an infinite-dimensional closed subspace with Schauder basis in which the canonical projections $(P_n)_n$ are bounded operators. Moreover, $\sup_n \|P_n\| < +\infty$.*

2.3 Spectral Theorem on inner product spaces

Let $T : N \rightarrow N$ be a compact self-adjoint operator, and $\widehat{T} : \widehat{N} \rightarrow \widehat{N}$ be the compact self-adjoint operator such that \widehat{T} is the bounded linear extension of T .

The following is the Spectral Theorem on inner product spaces.

Theorem 2.4. *Let N be an inner product space, $T : N \rightarrow N$ be a compact self-adjoint operator such that $T \neq 0$, $\text{sp}(T)$ be the spectrum of T , and $\text{sp}(\widehat{T})$ be the spectrum of \widehat{T} .*

- (i) *If $\text{sp}(T)$ is an infinite set such that $\text{sp}(T) \subsetneq \text{sp}(\widehat{T})$, then for each $x \in N$, there exists $w_x \in N$ such that $T(x) = \sum_{i=1}^{\infty} x_i \lambda_i \langle x, x_i \rangle + w_x$;*
- (ii) *If $\text{sp}(T)$ is an infinite set such that $\text{sp}(T) = \text{sp}(\widehat{T})$, then for each $x \in N$, $T(x) = \sum_{i=1}^{\infty} x_i \lambda_i \langle x, x_i \rangle$;*
- (iii) *If the cardinality of $\text{sp}(T)$ is a non-null natural number, then there exists a natural number $s \in \mathbb{N} \setminus \{0\}$ such that for all $x \in N$, $T(x) = \sum_{i=1}^s x_i \lambda_i \langle x, x_i \rangle$*

where (λ_i) 's are the eigenvalues of T .

3 Problems

It is known that every Banach space with Schauder basis is separable (see [13, Proposition 4.1.10]). Banach has questioned if the converse holds: Does every separable Banach space admits a Schauder basis? In [9], Enflo gave a negative answer to this question, exhibiting a separable Banach space that has no Schauder basis.

Problem 1. *(The basis problem for normed spaces) Let X be a separable normed space which is not a Banach one. Does X admits an essential Schauder basis?*

In the sequence, we introduce the definition of (essential) unconditional basis.

Definition 7. Let X be a Banach space, and $(x_n)_{n=1}^{\infty}$ a Schauder basis in X . The basis $(x_n)_{n=1}^{\infty}$ is an unconditional basis if, for each x in X , there exists a unique expansion of the form

$$x = \sum_{n=1}^{\infty} a_n x_n$$

where the sum converges unconditionally.

Gowers and Maurey in [11] showed that there exists a Banach space that do not contain unconditional basis.

Definition 8. Let X be a normed space, and $(x_n)_{n=1}^{\infty}$ an essential Schauder basis in X . The basis $(x_n)_{n=1}^{\infty}$ is an essential unconditional basis if, for each x in X , there exists a unique expansion of the form

$$x = \sum_{n=1}^{\infty} a_n x_n$$

where the sum converges unconditionally.

We end this section with the following question.

Problem 2. *(The unconditional basic sequence problem for normed spaces) Let X be a normed space which it is not a Banach space. Does X admits an essential unconditional basis?*

3.1 Organization of the text

In Sections 1, 2 and 3, we provide preliminary definitions in order to present the statements of the main results together with some applications and problems. In Section 4 we state some auxiliary results and prove some useful properties of Essential Schauder Basis. In Section 5, we give the proofs of applications of main results, divided into three subsections 5.1, 5.2 and 5.3, one for each of the applications. In Section 6, we give the proofs of our theorems, divided into two subsections 6.1 and 6.2, one for each of the Main Theorems A and B, respectively.

4 Auxilar Results

4.1 Essential Schauder basis

In the original Schauder's definition, there was the requirement that coordinates functional should be continuous. However, in [2, pag 111] is proved that this condition holds for Banach spaces. But, if the space is not complete, this assertion is false (see [5, Example 12.5]). We observe that if there exists an essential Schauder basis, then we recover this important property, as follows.

Theorem 4.1. *Each coordinate functional associated to an essential Schauder basis $(x_n)_{n=1}^\infty$ is a bounded linear application.*

Proof. Fix $n \in \mathbb{N}$, and let $x \in X$ be an arbitrary element, so we can write $x = \sum_{j=1}^\infty a_j x_j$. We are going to show that $x_n^* : X \rightarrow \mathbb{K}$ is a bounded linear application. In fact,

$$\begin{aligned} \|x_n\| \cdot |x_n^*(x)| &= \|x_n x_n^*(x)\| \\ &= \left\| \sum_{i=1}^n x_i^*(x) x_i - \sum_{i=1}^{n-1} x_i^*(x) x_i \right\| \\ &\leq \left\| \sum_{i=1}^n x_i^*(x) x_i \right\| + \left\| \sum_{i=1}^{n-1} x_i^*(x) x_i \right\| \\ &\leq 2\eta((a_j)_{j=1}^\infty) \leq 2\|(T_X)^{-1}\| \cdot \|x\|. \end{aligned}$$

□

Corollary 4.2. *Let $(x_n)_{n=1}^\infty$ be an essential Schauder basis for the normed space X . Then, for each $n \in \mathbb{N}$, the linear operator*

$$P_n : X \rightarrow X, P_n \left(\sum_{i=1}^\infty a_j x_j \right) = \sum_{i=1}^n a_j x_j$$

is bounded.

Proof. Note that $P_n(\cdot) = \sum_{i=1}^n x_i^*(\cdot) x_i$, so by Theorem (4.1), P_n is a bounded operator. □

In the proof of the next result, we are going to use some arguments from the proof of Corollary 4.1.17 of [13].

Theorem 4.3. *Let $(x_n)_{n=1}^\infty$ be an essential Schauder basis for a normed space X and $(P_n)_{n=1}^\infty$ the canonical projections. Then $\sup_n \|P_n\| < \infty$.*

Proof. For each x in X such that $\|x\| \leq 1$, we have that

$$\begin{aligned} \|P_n(x)\| &= \left\| \sum_{i=1}^n a_j x_j \right\| \leq \\ \sup_n \left\| \sum_{i=1}^n a_j x_j \right\| &= \eta((a_j)_{j=1}^\infty) \leq \\ \|(T_X)^{-1}\| \cdot \|x\| &\leq \|(T_X)^{-1}\|. \end{aligned}$$

So, $\|P_n\| \leq \|(T_X)^{-1}\|$ for all n in \mathbb{N} and $\sup_n \|P_n\| \leq \|(T_X)^{-1}\|$, and we are done. □

The number $K_{(x_n)_{n=1}^\infty} := \sup_n \|P_n\|$ is called *essential constant of basis* $(x_n)_{n=1}^\infty$. Note that $\|P_n\| \geq 1$ for all n , then $K_{(x_n)_{n=1}^\infty} \geq 1$.

Corollary 4.4. *Let $(x_n)_{n=1}^\infty$ be an essential Schauder basis for a normed space X and $(x_n^*)_{n=1}^\infty$ be the coefficient operators. Then, for each $k \in \mathbb{N}$,*

$$1 \leq \|x_k^*\| \cdot \|x_k\| \leq 2K_{(x_n)_{n=1}^\infty} \quad (4)$$

Proof. First, we note that

$$1 = x_k^*(x_k) = |x_k^*(x_k)| \leq \|x_k^*\| \cdot \|x_k\|. \quad (5)$$

Now, let x be an arbitrary element of X such that $x \neq 0$, so

$$\begin{aligned} \|x_k\| \cdot |x_k^*(x)| &= \|x_k x_k^*(x)\| \\ &= \left\| \sum_{i=1}^k x_i^*(x) x_i - \sum_{i=1}^{k-1} x_i^*(x) x_i \right\| \\ &\leq \left\| \sum_{i=1}^k x_i^*(x) x_i \right\| + \left\| \sum_{i=1}^{k-1} x_i^*(x) x_i \right\| \\ &= \|P_k(x)\| + \|P_{k-1}(x)\| \\ &\leq \|P_k\| \cdot \|x\| + \|P_{k-1}\| \cdot \|x\| \\ &\leq 2K_{(x_n)_{n=1}^\infty} \|x\|. \end{aligned}$$

This implies that

$$\|x_k\| \cdot \|x_k^*\| \leq 2K_{(x_n)_{n=1}^\infty}.$$

□

5 Proof of applications of main results

5.1 Proof of Bessaga-Pelczyński's Selection Principle

Using Theorem A, it is possible to show the following result.

Corollary 5.1. *Let $(x_n)_{n=1}^\infty$ be an essential basic sequence in a normed space X . Then $K_{(x_n)_{n=1}^\infty} = \inf\{M : M \text{ satisfies (2)}\}$.*

We are going to provide the definitions and required results to show the Bessaga-Pelczyński's Selection Principle for normed spaces.

Lemma 5.2. *Let $(x_n)_{n=1}^\infty$ be an essential Schauder's basis in a normed space X and $(y_n)_{n=1}^\infty$ be an essential block basic sequence relative to $(x_n)_{n=1}^\infty$. Then $(y_n)_{n=1}^\infty$ is an essential basic sequence in X and $K_{(y_n)_{n=1}^\infty} \leq K_{(x_n)_{n=1}^\infty}$.*

Proof. Use the Banach-Grunblum's criterion for normed spaces, Theorem A, and Corollary 5.1. \square

Following the proof given in Theorem 4.3.19 of [13] for Banach spaces, and using Corollary 4.4, Lemma 5.2 and Theorem B we obtain the Bessaga-Pelczyński's selection principle for normed spaces, Theorem 2.1.

In the proof of Corollary 2.2, we use the arguments from the proof of Corollary 10.4.9 of [4].

Proof of Corollary 2.2. Consider the subspace $\overline{[y_n : n \in \mathbb{N}]}$ of X , that is separable. By Banach-Mazur's Theorem, $\overline{[y_n : n \in \mathbb{N}]}$ is isometrically isomorphic to a subspace of $C[0, 1]$, so there exists a linear isometry $T : \overline{[y_n : n \in \mathbb{N}]} \rightarrow C[0, 1]$ such that $\overline{[y_n : n \in \mathbb{N}]}$ and $T(\overline{[y_n : n \in \mathbb{N}]})$ are isometrically isomorphic. Observe that $\inf_n \|T(y_n)\| = \inf_n \|y_n\| > 0$. Let $(x_n)_{n=1}^\infty$ be a Schauder basis for $C[0, 1]$, and $(x_n^*)_{n=1}^\infty$ be the coefficient operators. By boundedness of operator T , since $(y_n)_n$ converges to 0 in the weak topology, we obtain that $(T(y_n))_n$ converges to 0 in the weak topology. So

$$\lim_{n \rightarrow \infty} x_k^*(T y_n) = 0 \text{ for all } k \text{ in } \mathbb{N}.$$

By Bessaga-Pelczyński's selection principle for normed spaces (Theorem 2.1), there exists an essential basic subsequence $(T(y_{n_k}))_{k=1}^\infty$ of $(T(y_n))_n$. Using that $T^{-1} : \overline{[y_n : n \in \mathbb{N}]} \rightarrow \overline{[y_n : n \in \mathbb{N}]}$ is an isometric isomorphism, we have that $(y_{n_k})_{k=1}^\infty = (T^{-1}(T(y_{n_k})))_{k=1}^\infty$ is an essential basic sequence. This concludes the proof of Corollary 2.2. \square

5.2 Proof of Banach problem for normed spaces

Proof of Theorem 2.3. We will use the same sequence $(x_n)_{n=1}^\infty$ obtained in the traditional

proof (see e.g. [7, 8]). By Banach-Grunblum's criterion for normed spaces, Theorem A, we obtain that $(x_n)_{n=1}^\infty$ is an essential basic sequence. In particular, $(x_n)_{n=1}^\infty$ is an essential Schauder basis. Then, by Theorem (4.3), the canonical projections $(P_n)_n$ are bounded and $\sup_n \|P_n\| < +\infty$. \square

5.3 Proof of Spectral Theorem

We begin by proving a consequence of Theorem A.

Lemma 5.3. *Let N be an inner product space, $S = \{x_n : n \in \mathbb{N}\}$ be an orthonormal set of N . Then*

- (i) *For all $x \in N$, and any $n \in \mathbb{N}$, we have that $\sum_{i=1}^n |\langle x, x_i \rangle|^2 \leq \|x\|^2$ (and then $\sum_{i=1}^\infty |\langle x, x_i \rangle|^2 \leq \|x\|^2$);*
- (ii) *$(x_n)_{n=1}^\infty$ is an essential Schauder basis for $\widehat{[x_n : n \in \mathbb{N}]} \subseteq \widehat{N}$;*
- (iii) *$(x_n)_{n=1}^\infty$ is an essential Schauder basis for $\overline{[x_n : n \in \mathbb{N}]} \subseteq N$.*

Proof. Just note that $0 \leq \langle x - \sum_{i=1}^n \langle x, x_i \rangle x_i, x - \sum_{i=1}^n \langle x, x_i \rangle x_i \rangle$ to prove item (i). To obtain items (ii) and (iii), by Banach-Grunblum's criterion for normed spaces, Theorem (A), we are reduced to prove that for all sequence of scalar $(a_n)_{n=1}^\infty$ we have that $\left\| \sum_{i=1}^m a_i x_i \right\| \leq \left\| \sum_{i=1}^n a_i x_i \right\|$ whenever $n \geq m$. But it is clear by orthonormality of S . \square

We need these auxiliar results.

Lemma 5.4. *Let N be an inner product space and $T : N \rightarrow N$ be a compact self-adjoint operator. Then $\widehat{T} : \widehat{N} \rightarrow \widehat{N}$ is a compact self-adjoint operator where \widehat{T} is the bounded linear extension of T and \widehat{N} is the completion of N such that N is dense in \widehat{N} .*

Proof. The proof is straightforward. \square

Lemma 5.5. *Let N be an inner product space and $U : N \rightarrow N$ be a compact self-adjoint operator. If U is not the null-operator, there exist $x \in N \setminus \{0\}$ and $\lambda \in \mathbb{R} \setminus \{0\}$ such that $U(x) = \lambda x$.*

Proof. Use the same arguments present in the proof of this result for compact self-adjoint operators on Hilbert spaces. \square

Proof of Theorem 2.4. By Lemma 5.4, $\widehat{T} : \widehat{N} \rightarrow \widehat{N}$ is a compact self-adjoint operator on a Hilbert space. There exist $\alpha_i \in \mathbb{R} \setminus \{0\}$ eigenvalues of \widehat{T} for i in a not empty subset A of \mathbb{N} such that $\ker(\widehat{T} - \alpha_i \widehat{I})$ is a finite dimensional subspace where $I : N \rightarrow N$ is given by $I(x) = x$ and \widehat{I} is the extension of I to \widehat{N} . For each $n \in A$, note that $N \cap \ker(\widehat{T} - \alpha_n \widehat{I}) = \ker(T - \alpha_n I) \subseteq N$, define $t_n := \dim \ker(T - \alpha_n I)$ and $\widehat{t}_n := \dim \ker(\widehat{T} - \alpha_n \widehat{I})$. Let be J be the subset of \mathbb{N} given by $\{n \in A : N \cap \ker(\widehat{T} - \alpha_n \widehat{I}) \neq \{0\}\}$, we have that $1 \leq t_n \leq \widehat{t}_n$. So for each $n \in J$ there exist $v_{n,1}, \dots, v_{n,t_n}$ such that $T(v_{n,j}) = \alpha_n v_{n,j}$ for all $j \in \{1, \dots, t_n\}$ and $\{v_{n,j} : 1 \leq j \leq t_n\}$ is an orthonormal set. Then $S = \{v_{n,j} : n \in J \text{ and } j \in \{1, \dots, t_n\}\}$ is an orthonormal set of N . We may write the orthonormal set S as $\{x_\ell \in N : \ell \in \mathcal{J}\}$ where \mathcal{J} is a subset of \mathbb{N} , and $T(x_\ell) = \lambda_\ell x_\ell$ for each $\ell \in \mathcal{J}$.

For each $n \in A$, there exists an orthonormal set $\{v_{n,t_n+1}, \dots, v_{n,\widehat{t}_n}\}$ of $\ker(\widehat{T} - \alpha_n \widehat{I})$, and then $\{v_{n,1}, \dots, v_{n,\widehat{t}_n}\}$ is an orthonormal set of $\ker(\widehat{T} - \alpha_n \widehat{I})$. Then $R = \{v_{n,j} : n \in A \text{ and } j \in \{t_n + 1, \dots, \widehat{t}_n\}\}$ is an orthonormal set of \widehat{N} . We may write the orthonormal set R as $\{y_\ell \in N : \ell \in \widetilde{A}\}$ where \widetilde{A} is a subset of \mathbb{N} , and $\widehat{T}(y_\ell) = \beta_\ell y_\ell$ for each $\ell \in \widetilde{A}$.

Suppose that $\text{sp}(T)$ is an infinite set, then $\mathcal{J} = \mathbb{N}$. We are going to prove items (i) and (ii).

Let \widehat{F} be the completion of $F = [x_\ell : \ell \in \mathcal{J}]$, \widehat{G} be the completion of $G = [y_\ell : \ell \in \widetilde{A}]$, \widehat{E} be the completion of $E = \left[\bigcup_{n=1}^{\infty} \ker(\widehat{T} - \alpha_n \widehat{I}) \right] = [F \cup G]$. From the Spectral Theorem for Hilbert Spaces, we have that $\widehat{N} = \widehat{E} \oplus \ker(\widehat{T})$, and observe that $\widehat{E} = \widehat{F} \oplus \widehat{G}$.

Since S is an orthonormal set of N , from Lemma 5.3, we have that $(x_\ell)_{\ell=1}^{\infty}$ is an essential Schauder basis for \widehat{F} . For any $a \in \widehat{F}$, there exists a sequence of scalar $(\xi_n)_{n=1}^{\infty}$ such

that $a = \sum_{i=1}^{\infty} \xi_i x_i$ where $\xi_i = \langle a, x_i \rangle$ for each $i \in \mathbb{N}$, so $\widehat{T}(a) = \sum_{i=1}^{\infty} x_i \lambda_i \langle a, x_i \rangle$.

Since R is an orthonormal set of \widehat{N} , from Lemma 5.3, we have that $(y_\ell)_{\ell \in \widetilde{A}}$ is an essential Schauder basis for \widehat{G} . For any $b \in \widehat{G}$, there exists a sequence of scalar $(\gamma_n)_{n \in \widetilde{A}}$ such that $b = \sum_{i \in \widetilde{A}} \gamma_i y_i$ where $\gamma_i = \langle b, y_i \rangle$ for each $i \in \widetilde{A}$, so $\widehat{T}(b) = \sum_{i \in \widetilde{A}} y_i \beta_i \langle b, y_i \rangle$.

We need the following result.

Lemma 5.6. $\widehat{T}(\widehat{F}) \subseteq N \cap \widehat{F}$.

Proof of Lemma 5.6. Let x be an arbitrary element of \widehat{F} . Since S is an orthonormal set of N , from Lemma 5.3, we have that $(x_\ell)_{\ell=1}^{\infty}$ is an essential Schauder basis for \widehat{F} . There exists a sequence of scalar $(a_n)_{n=1}^{\infty}$ such that $x = \sum_{i=1}^{\infty} a_i x_i$ where $a_i = \langle x, x_i \rangle$ for each $i \in \widetilde{J}$.

Note that $\widehat{T}(x) = \sum_{i=1}^{\infty} a_i \lambda_i x_i =$

$\lim_{n \rightarrow \infty} \sum_{i=1}^n a_i \lambda_i x_i$, so $\widehat{T}(x) \in \widehat{F}$.

Consider the sequence $(y_n)_{n=1}^{\infty}$ where $y_n = \sum_{i=1}^n x_i a_i \in N$ for each $n \in \mathbb{N}$. Using

Lemma 5.3, we see that $\|y_n\|^2 = \sum_{i=1}^n (a_i)^2 =$

$\sum_{i=1}^n |\langle x, x_i \rangle|^2 \leq \|x\|^2$. Then $(y_n)_{n=1}^{\infty}$ is a

bounded sequence of N , by compactness of T , there exists a subsequence $(T(y_{n_k}))_{k=1}^{\infty}$ that converges to some point $y \in N$. We have that y_n converges to x , and then $\widehat{T}(y_n)$ converges to $\widehat{T}(x)$, so $\widehat{T}(x) = y \in N$. This completes the proof of Lemma 5.6. \square

So for each $x \in \widehat{N}$ there exist $a_x \in \widehat{F}$, $b_x \in \widehat{G}$ and $c_x \in \ker(\widehat{T})$ such that $x = a_x + b_x + c_x$, and then $\widehat{T}(x) = \widehat{T}(a_x) + \widehat{T}(b_x)$ where $\widehat{T}(a_x) = \sum_{i=1}^{\infty} x_i \lambda_i \langle a_x, x_i \rangle$ and $\widehat{T}(b_x) = \sum_{i \in \widetilde{A}} y_i \beta_i \langle b_x, y_i \rangle$. By Lemma 5.6, we get that $\widehat{T}(a_x) \in N$.

Now, suppose that $x \in N$, so $T(x) = \widehat{T}(x) \in N$ and $\widehat{T}(x) = \widehat{T}(a_x) + \widehat{T}(b_x)$ with $\widehat{T}(a_x) \in N$, then $\widehat{T}(b_x) \in N$. Note that $\langle a_x, x_i \rangle = \langle x, x_i \rangle$ and $\langle b_x, y_i \rangle = \langle x, y_i \rangle$. For $x \in N$, we obtain that

$$T(x) = \sum_{i=1}^{\infty} x_i \lambda_i \langle x, x_i \rangle + \sum_{i \in \tilde{A}} y_i \beta_i \langle x, y_i \rangle$$

where $\sum_{i=1}^{\infty} x_i \lambda_i \langle x, x_i \rangle, \sum_{i \in \tilde{A}} y_i \beta_i \langle x, y_i \rangle \in N$.

This proves item (i).

Suppose that $\text{sp}(T) = \text{sp}(\widehat{T})$, this implies that $\tilde{A} = \emptyset$, and then $T(x) = \sum_{i=1}^{\infty} x_i \lambda_i \langle x, x_i \rangle$

for all $x \in N$. The item (ii) is proved.

It remains to show item (iii). Now, $\text{sp}(T)$ is a natural number k . This implies that $S = \{v_{n,j} : n \in J \text{ and } j \in \{1, \dots, t_n\}\}$ is a finite orthonormal set of N such that $T(v_{n,j}) = \alpha_n v_{n,j}$ for all $j \in \{1, \dots, t_n\}$ for each $n \in J$. We may write the orthonormal set S as $\{x_\ell \in N : \ell \in \{1, \dots, s\}\}$ for some $s \in \mathbb{N} \setminus \{0\}$ where $T(x_\ell) = \lambda_\ell x_\ell$ for each $\ell \in \{1, \dots, s\}$.

Note that $F = [x_\ell : \ell \in \{1, \dots, s\}]$ is a Banach space since F is a finite dimensional space. This implies that $N = F \oplus F^\perp$ where $F^\perp = \{w \in N : \langle w, z \rangle = 0 \text{ for all } z \in F\}$.

We claim that $F^\perp = \ker(T)$. A trivial verification shows that $\ker(T) \subseteq F^\perp$. Using that $T(F) \subseteq F$, we have that $T(F^\perp) \subseteq F^\perp$. Define $U = T|_{F^\perp} : F^\perp \rightarrow F^\perp$, and note that U is a compact self-adjoint operator. Suppose that F^\perp is not contained in $\ker(T)$, there exists $q \in F^\perp$ such that $T(q) \neq 0$, then $U(q) = T(q) \neq 0$ and U is not null-operator. By Lemma 5.5, there exist $\lambda \in \mathbb{R}$ and $p \in F^\perp \setminus \{0\}$ such that $T(p) = U(p) = \lambda p$, so $p \in F$. We obtain that $p \in F \cap F^\perp$, so $p = 0$. This contradiction shows that U is the null operator, and the claim is proved.

For $x \in N = F \oplus \ker(T)$, there exist $a_x \in F$ and $b_x \in \ker(T)$ such that $x = a_x + b_x$. Note that for $a \in F$, we have that $a = \sum_{i=1}^s \langle a, x_i \rangle x_i$ and $T(a) = \sum_{i=1}^s \langle a, x_i \rangle \lambda_i x_i$. We obtain that $T(x) = T(a_x) = \sum_{i=1}^s \langle x, x_i \rangle \lambda_i x_i$ since $\langle a_x, x_i \rangle = \langle x, x_i \rangle$ for all $i \in \{1, \dots, s\}$, and we are

done. □

6 Proofs of main results

6.1 The Banach-Grunblum's criterion for normed spaces

Proof of Theorem A. It is clear that (i) implies (ii). We are going to show that (ii) proves (iii).

Suppose that $(x_n)_{n=1}^{\infty}$ is an essential Schauder basis for $[x_n : n \in \mathbb{N}]$. Now, consider the canonical projections $(P_n)_{n=1}^{\infty}$ in $E = [x_n : n \in \mathbb{N}]$. By Theorem (4.3), we know that

$$1 \leq K_{(x_n)_{n=1}^{\infty}} = \sup_n \|P_n\| < \infty$$

Given a sequence of scalar $(a_n)_{n=1}^{\infty}$, if $n \geq m$, then

$$\begin{aligned} & \left\| \sum_{i=1}^m a_i x_i \right\| \\ &= \left\| P_m \left(\sum_{i=1}^n a_i x_i \right) \right\| \\ &\leq \|P_m\| \cdot \left\| \sum_{i=1}^n a_i x_i \right\| \quad (6) \\ &\leq K_{(x_n)_{n=1}^{\infty}} \left\| \sum_{i=1}^n a_i x_i \right\|. \end{aligned}$$

We are reduced to proving (i) from (iii). Suppose that (2) holds for $M \geq 1$. A straightforward calculation shows that the set $\{x_\ell : \ell \in \mathbb{N}\}$ is linearly independent.

For each $n \in \mathbb{N}$ consider the linear functional given by

$$\varphi_n : [x_\ell : \ell \in \mathbb{N}] \rightarrow \mathbb{K}, \varphi_n \left(\sum_{i=1}^k a_i x_i \right) = a_n$$

and the linear operator

$$\begin{aligned} T_n : [x_\ell : \ell \in \mathbb{N}] &\rightarrow [x_\ell : \ell \in \mathbb{N}], \\ T_n \left(\sum_{i=1}^k a_i x_i \right) &= \sum_{i=1}^n a_i x_i \end{aligned}$$

defining $a_{k+1} = \dots = a_n = 0$ if necessary.

It is clear that φ_n is a bounded linear operator for all n .

By (2), we have that

$$\left\| T_n \left(\sum_{i=1}^k a_i x_i \right) \right\| = \left\| \sum_{i=1}^n a_i x_i \right\| \leq M \left\| \sum_{i=1}^k a_i x_i \right\|$$

and T_n is bounded with $\|T_n\| \leq M$.

There exists a bounded linear extension $\Phi_n : \widehat{F} \rightarrow \mathbb{K}$ such that $\Phi_n|_F = \varphi_n$ and $\|\Phi_n\| = \|\varphi_n\|$ where $F = [x_\ell : \ell \in \mathbb{N}]$.

Consider the bounded linear operator $T_n : F \rightarrow F$, so we may consider this bounded linear operator $T_n : [x_\ell : \ell \in \mathbb{N}] \rightarrow \widehat{F}$. There exists a bounded linear extension $R_n : \widehat{F} \rightarrow \widehat{F}$ such that $R_n|_F = T_n$ and $\|R_n\| = \|T_n\|$.

Note that $T_n(z) = \sum_{i=1}^n a_i x_i = \sum_{i=1}^n \varphi_i(z) x_i$ for all $z \in F$.

For all $x \in \widehat{F}$ we have that

$$R_n(x) = \sum_{i=1}^n \Phi_i(x) x_i. \quad (7)$$

In fact, let $x \in \widehat{F}$ be an arbitrary element, so $x = \lim_{k \rightarrow \infty} y_k$ where $y_k \in [x_\ell : \ell \in \mathbb{N}]$. But R_n is a bounded linear operator, so

$$\begin{aligned} R_n(x) &= \lim_{k \rightarrow \infty} R_n(y_k) = \\ \lim_{k \rightarrow \infty} \sum_{i=1}^n \varphi_i(y_k) x_i &= \lim_{k \rightarrow \infty} \sum_{i=1}^n \Phi_i(y_k) x_i = \\ \sum_{i=1}^n \Phi_i(x) x_i. \end{aligned}$$

We obtain that $R_n(x) = \sum_{i=1}^n \Phi_i(x) x_i$ for all x in \widehat{F} . Now, given $x \in \widehat{F}$ and $\varepsilon > 0$, there exists $y = \sum_{j=1}^m a_j x_j \in [x_\ell : \ell \in \mathbb{N}]$ for some $m \geq 1$ such that $\|x - y\| < \varepsilon$. For $n > m$, we have

$$\begin{aligned} &\|x - R_n(x)\| \\ &\leq \|x - y\| + \|R_n(y) - y\| \\ &\quad + \|R_n(x) - R_n(y)\| \\ &\leq \|x - y\| + \|y - y\| \\ &+ \|R_n\| \cdot \|x - y\| \leq (1 + M)\varepsilon. \end{aligned}$$

Then $x = \lim_{n \rightarrow \infty} R_n(x)$. Using (7), we obtain

$$x = \lim_{n \rightarrow \infty} R_n(x) = \lim_{n \rightarrow \infty} \sum_{i=1}^n \Phi_i(x) x_i = \sum_{i=1}^{\infty} \Phi_i(x) x_i.$$

The uniqueness of the above representation is clear. So $(x_n)_{n=1}^{\infty}$ is a Schauder basis for \widehat{F} .

Given $x \in \widehat{F}$ with $x = \sum_{i=1}^{\infty} a_i x_i$. By (2), for each $n \in \mathbb{N}$ we have that

$$\left\| \sum_{i=1}^n a_i x_i \right\| \leq M \left\| \sum_{i=1}^{\infty} a_i x_i \right\| = M \|x\|. \quad (8)$$

Then

$$\sup_n \left\| \sum_{i=1}^n a_i x_i \right\| \leq M \left\| \sum_{i=1}^{\infty} a_i x_i \right\| = M \|x\|. \quad (9)$$

and note that

$$\begin{aligned} \|x\| &= \left\| \sum_{i=1}^{\infty} a_i x_i \right\| \\ &= \lim_{n \rightarrow \infty} \left\| \sum_{i=1}^n a_i x_i \right\| \leq \sup_n \left\| \sum_{i=1}^n a_i x_i \right\|. \end{aligned} \quad (10)$$

Using (9) and (10), we obtain that $T_{\widehat{F}} : \mathcal{L}_{\widehat{F}} \rightarrow \widehat{F}$ given by $T_{\widehat{F}}((a_n)_{n=1}^{\infty}) = \sum_{n=1}^{\infty} a_n x_n$ is a linear isomorphism. Then $(x_n)_{n=1}^{\infty}$ is an essential Schauder basis for \widehat{F} , and we are done. \square

6.2 The Bessaga-Pelczyński Theorem for normed spaces

Proof of Theorem B. Given a sequence $(a_n)_{n=1}^{\infty}$ of scalars,

$$\begin{aligned} & \left\| \sum_{i=1}^n a_i(x_i - y_i) \right\| \\ &= \left\| \sum_{i=1}^n x_i^* \left(\sum_{j=1}^n a_j x_j \right) (x_i - y_i) \right\| \\ &\leq \sum_{i=1}^n |x_i^* \left(\sum_{j=1}^n a_j x_j \right)| \cdot \|x_i - y_i\| \\ &\leq \left\| \sum_{j=1}^n a_j x_j \right\| \left(\sum_{i=1}^n \|x_i^*\| \cdot \|x_i - y_i\| \right) \\ &\leq \lambda \left\| \sum_{i=1}^n a_i x_i \right\|, \end{aligned}$$

we obtain that

$$\begin{aligned} & \left| \left\| \sum_{i=1}^n a_i x_i \right\| - \left\| \sum_{i=1}^n a_i y_i \right\| \right| \leq \\ & \left\| \sum_{i=1}^n a_i x_i - \sum_{i=1}^n a_i y_i \right\| \leq \lambda \left\| \sum_{i=1}^n a_i x_i \right\|. \end{aligned}$$

So

$$\begin{aligned} & (1 - \lambda) \left\| \sum_{i=1}^n a_i x_i \right\| \\ & \leq \left\| \sum_{i=1}^n a_i y_i \right\| \quad (11) \\ & \leq (1 + \lambda) \left\| \sum_{i=1}^n a_i x_i \right\|, \end{aligned}$$

for all n in \mathbb{N} . But $(x_n)_{n=1}^\infty$ is an essential basic sequence in X . By Banach-Grünblum's criterion for normed spaces, Theorem (A), there exists $M \geq 1$ such that if $m \geq n$ then

$$\begin{aligned} & \left\| \sum_{i=1}^n a_i y_i \right\| \\ & \leq (1 + \lambda) \left\| \sum_{i=1}^n a_i x_i \right\| \quad (12) \\ & \leq (1 + \lambda) M \left\| \sum_{i=1}^m a_i x_i \right\| \\ & \leq \frac{(1 + \lambda) M}{(1 - \lambda)} \left\| \sum_{i=1}^m a_i y_i \right\|. \end{aligned}$$

From Banach-Grünblum's criterion for normed spaces, Theorem (A), $(y_n)_{n=1}^\infty$ is an essential basic sequence in X .

We are going to show that the series $\sum_{n=1}^\infty a_n y_n$ converges if the series $\sum_{n=1}^\infty a_n x_n$ is convergent.

Define $F = [x_n : n \in \mathbb{N}]$ and $Y = [y_n : n \in \mathbb{N}]$. Consider $T : F \rightarrow Y$ the linear operator given by $T \left(\sum_{i=0}^k x_i a_i \right) = \sum_{i=0}^k y_i a_i$ for $x = \sum_{i=0}^k x_i a_i$ in F . Note that if $x \neq 0$, we

$$\text{have that } \left\| T \left(\frac{x}{\|x\|} \right) \right\| = \frac{\|T(x)\|}{\|x\|} = \frac{\left\| \sum_{i=1}^k a_i y_i \right\|}{\left\| \sum_{i=1}^k a_i x_i \right\|}$$

for $x = \sum_{i=0}^k x_i a_i$ in F , and by (11) we obtain that $\left\| T \left(\frac{x}{\|x\|} \right) \right\| \leq (1 + \lambda)$.

So $T : F \rightarrow Y$ is a bounded linear operator. We may consider T as the following bounded operator $T : \overline{F} \subseteq X \rightarrow Y$. There exists a bounded linear extension $\widehat{T} : \widehat{F} \rightarrow \widehat{Y}$ such that $\widehat{T}|_{\overline{[x_n : n \in \mathbb{N}]}} = T$ with $\|\widehat{T}\| = \|T\|$.

Now, suppose that $x = \sum_{n=1}^\infty a_n x_n \in \overline{F} \subseteq X$ is convergent. This implies that

$$\begin{aligned} T(x) &= \widehat{T}(x) = \widehat{T} \left(\lim_{n \rightarrow \infty} \sum_{i=1}^n a_i x_i \right) = \\ & \lim_{n \rightarrow \infty} \widehat{T} \left(\sum_{i=1}^n a_i x_i \right) = \lim_{n \rightarrow \infty} \sum_{i=1}^n a_i y_i, \end{aligned}$$

then the series $T(x) = \widehat{T}(x) = \sum_{i=1}^\infty a_i y_i \in Y \subseteq X$ is convergent.

Analogously, we can show that if the $\sum_{i=1}^\infty a_i y_i$ is convergent, then $\sum_{i=1}^\infty a_i x_i$ is convergent. \square

7 Conclusion

We give a generalization of the well known criterion of Banach-Grünblum and the Bessaga-Pełczyński Theorem in normed spaces setting (not necessarily Banach one). As application, we show the Principle of Selection of Bessaga-Pełczyński for normed

spaces and the Spectral Theorem for compact self-adjoint operators on inner product spaces.

In the case of Banach spaces (normed and complete), this theory can be applied to solve differential equations with a low computational cost [15]. This is an important issue in applied sciences, because many problems can be formulated through differential/integral equations. Although our work in this paper is not specifically on applied science, we expect that our generalization of this theory to spaces beyond Banach ones help to improve its application. Also, in it has been developed a method, by using the least-squares method and a Schauder basis [14], which provides a numerical solution for a wide class of linear differential or integral equations. The least squares method is a statistical technique used to find the line of best fit for a given set of data points. This method minimizes the sum of the squares of the residuals, which are the differences between the observed values and the values predicted by the model. We hope that our techniques help to improve these results and be useful to applied sciences in some moment.

Acknowledgment:

No fee has been paid for this publication. The authors are grateful to Instituto de Matemática da Universidade Federal da Bahia for the hospitality in preparation of this paper.

References:

- [1] F. Albiac, N.J. Kalton. Topics in Banach Space Theory. *Graduate Texts in Mathematics*, 233. New York: Springer-Verlag, 2006.
- [2] S. Banach. Théorie des opérations linéaires. Monografie Matematyczne 1. Warsaw, 1932.
- [3] C. Bessaga; A. Pełczyński. On bases and unconditional convergence of series in Banach spaces. *Studia Mathematica (1958)*. Volume: 17, Issue: 2, page 151-164.
- [4] G. Botelho, D. Pellegrino, E. Teixeira. Fundamentos de análise funcional. *Colecao textos universitarios*. Rio de Janeiro: SBM, 2015.
- [5] C. Swartz. Elementary Functional Analysis. *World Scientific*. Singapore, 2010.
- [6] M.M. Day. On the Basis Problem in Normed Spaces. *Proceedings of the American Mathematical Society*. Vol. 13, No. 4 (Aug., 1962), pp. 655-658.
- [7] M. Daws. Introduction to Bases in Banach Spaces, 2005.
www1.maths.leeds.ac.uk/mdaws/pubs/bases.pdf
- [8] J. Diestel. Sequences and Series in Banach Spaces. *Graduate texts in mathematics; 92* New York: Springer-Verlag, 1984.
- [9] P. Enflo. A counterexample to the approximation problem in Banach spaces. *Acta Mathematica*. 130 (1): 309-317, 1973.
- [10] B. R. Gelbaum. Notes on Banach spaces and bases. *An. Acad. Bras. Cienc.* 30 (1958), 29-36.
- [11] W.T. Gowers, B. Maurey. The unconditional basic sequence problem. arXiv:math/9205204, 1992.
- [12] C. W. McArthur. Developments in Schauder basis theory. *Bull. Amer. Math. Soc.* 78 (1972), 877-908.
- [13] R.E. Megginson. An introduction do Banach space theory. Springer, 1998.
- [14] A. Palomares, M. Pasadas, V. Ramírez, M. Ruiz Galán. A convergence result for a least-squares method using Schauder bases. *Mathematics and Computers in Simulation* 77 (2008) 274–28.
- [15] A. Palomares, M. Ruiz Galán. Isomorphisms, Schauder bases in Banach spaces and numerical solution of integral and differential equations. *Numerical Functional Analysis and Optimization*, 26(1):129–137, 2005.
- [16] J. Schauder. Theorie stetiger Abbildungen in Funktionalräumen. *Math. Z.*, 26, 47-65, 1927.

- [17] J. Schauder. Eine eigenschaft des haarschen orthogonalsystems. *Math. Z.*, 28, 317-320, 1928.
- [18] D. Varsamis, A. Kamilali. Calculation of Determinant of a two-variable Polynomial Matrix in Complex Basis. *WSEAS Transactions on systems and control*, (2022) Vol.17. DOI: 10.37394/23203.2022.17.44
- [19] M. Yaremenko. Trace class in separable reflexive Banach spaces, Lidskii theorem. *Equations, an International Journal of Mathematical and Computational Methods in Science and Engineering*, vol.2, 19, 2022. E-ISSN: 2732-9976 DOI:10.37394/232021.2022.2.19

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

V.C. was supported by CAPES-Coordenação de Aperfeiçoamento de Pessoal de Nível Superior(Brazil). L.S. is partially supported by FAPERJ-Funda,ção Carlos Chagas Filho de Amparo à Pesquisa do Estado do Rio de Janeiro Projects APQ1-E-26/211.690/2021 SEI-260003/015270/2021 and JCNE-E-26/200.271/2023 SEI-260003/000640/2023, by CAPES — Finance Code 001, CNPq-Conselho Nacional de Desenvolvimento Científico e Tecnológico Projeto Universal 404943/2023-3.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

Creative Commons Attribution License 4.0 (Attribution 4.0 International, CC BY 4.0)

This article is published under the terms of the Creative Commons Attribution License 4.0

https://creativecommons.org/licenses/by/4.0/deed.en_US