Control of the Motion of an Inverted Spherical Pendulum on a Moving Base. Hybrid Impact Approach

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Abstract: -A new hybrid method for the construction of control actions of a linear control system with constant coefficients is considered in this paper. It is assumed in this paper that a part of the discussed system meets some conditions. Some states of the main system are considered to be control actions for a subsystem for which and LQR stabilizer is acquired. Then, those control actions of the subsystem are used to construct the control actions for the main system. In the problem of controlling the motion of a complex linear system of an inverted spherical pendulum on a moving base, a new approach to the construction of control actions (hybrid action method) was used. It is assumed that a component of the complex system under discussion satisfies certain conditions. The inertial forces at the center of mass of the base of the composite system are considered to be the controlling influences on the inversion of the pendulum, for which the LQR stabilizer was purchased. The determined internal control actions on the inverted pendulum are then used to construct external control actions on the base of the composite system. In the end, a numerical analysis was carried out.

Key-words: —control problem, hybrid control, linear system, optimal solutions, optimal stabilization, control actions, numerical example.

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1. Introduction

With the development of modern technology, many scientists are studying a wide variety of movements of complex systems, as well as control of the movements of linear systems subjected to complex external actions. Similar studies are carried out both in the field of motion control of composite linear systems, and in the field of motion control of systems with heteronode influences (hybrid actions). Some of those problems are addressed as hybrid control problems. In [1], [2] the authors present a method to achieve such a hybrid control of position and force. In [3] the hybrid systems as causal and consistent dynamical systems were discussed, and a general formulation for an optimal hybrid control problem was proposed. In [4] the authors survey recent results in the field of optimal control of hybrid and switched systems. They first summarize results that use different problem formulations and then explore the underlying relations among them. Specifically, based on the type of switching, they focus on two important classes of problems: internally forced switching (IFS) problems and externally forced switching (EFS) problems. For IFS problems, they focus on optimal control techniques for piecewise affine systems. For EFS problems, methodologies of two-stage optimization, embedding transformation, and switching LQR design are investigated. Detailed optimization methods found in the literature are discussed.

Research on hybrid control problems has also been done for the study of the Covid-19 epidemic. In [5] is being study epidemics using mathematical modeling, which is crucial for understanding its dynamics and proposing potential control measures. A generalized epidemiological model corresponding to a pandemic is proposed, in which its dynamics are presented as a new hybrid system obtained by combining a deterministic model with a stochastic model.

2. Problem Description

A new hybrid method for construction of control actions of a linear control system with constant coefficients is considered in paper [6]. Let as start by presenting some of the main points of the work.

Assume we have a state space model which have the following dynamics

$$\dot{x}_i = a_{i1}x_1 + \cdots + a_{in}x_n + p_{i1}y_1 + \cdots + \\ + p_{ik}y_k + b_{i1}u_1 + \cdots + b_{ir}u_r, \quad (1)$$

$$\dot{y}_j = c_{j1}y_1 + \cdots + c_{jk}y_k + d_{j1}x_1 + \cdots + d_{jm}x_m, \quad (2)$$

where the coefficients a_{il}, b_{is}, c_{jq}, d_{jf}, p_{iq} are real constants and $i = 1, \dots, n$, $j = 1, \dots, k$, $r \leqslant n - m$, $l = 1, \dots, n$, $m \leqslant k \leqslant n$, $s = 1, \dots, r$, $q = 1, \dots, k$. Also $x_1, \dots, x_n, y_1 \dots, y_k$ are the states of the system, and $u_1, \dots, u_r$ is the control actions applied to the system. We can rewrite the system (1)-(2) as a system of matrix equations

$$\dot{x} = Ax + Py + Bu, \quad (3)$$
$$\dot{y} = Cy + D\bar{x}. \quad (4)$$

where

$$A=\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \vdots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}, \quad P=\begin{pmatrix} p_{11} & \cdots & p_{1k} \\ \vdots & \vdots & \vdots \\ p_{n1} & \cdots & p_{nk} \end{pmatrix},$$

$$B=\begin{pmatrix} b_{11} & \cdots & b_{1r} \\ \vdots & \vdots & \vdots \\ b_{n1} & \cdots & b_{nr} \end{pmatrix}, \quad C=\begin{pmatrix} c_{11} & \cdots & c_{1k} \\ \vdots & \vdots & \vdots \\ c_{k1} & \cdots & c_{kk} \end{pmatrix},$$

$$D=\begin{pmatrix} d_{11} & \cdots & d_{1m} \\ \vdots & \vdots & \vdots \\ d_{k1} & \cdots & d_{km} \end{pmatrix}.$$

Here, $x=(x_1\cdots x_n)^T$ is an n dimensional column vector, $y=(y_1\cdots y_k)^T$ is a k dimensional column vector, $\bar{x}=(\bar{x}_1\cdots\bar{x}_m)^T$ is an m dimensional column vector that contains some m states of x and $u=(u_1\cdots u_r)^T$ is an r dimensional column vector.

Let us now define the following problem.

Problem Definition 1. We are given the system (1)-(2) (or (3)-(4), the time period $[t_0,t_1]$, the initial position of some of the states (maximum number of the states of (3) can be $n/2$ and for (4) the number can be k of the system $(x(t_0);y(t_0))=(x_0;y_0)$ and the desired final position of some of the states (maximum number of the states of (3) can be $n/2$ and for (4) the number can be k of the system $x(t_1)=x_1$. It is required to find the control inputs $u(t)$, $(t_0\leqslant t\leqslant t_1)$ such that it drives the system from its given initial position to its desired final position.

Assume, the matrices A, B, C, D are such that

$$\operatorname{rank}K_1=\{D,CD,\ldots,C^{k-1}D\}=k \qquad (5)$$

and

$$\operatorname{rank}K_2=\{B_1,A_1B_1,\ldots,A_1^{n+k-1}B_1\}=n+k. \qquad (6)$$

Where A_1 is the following $(n+k)\times(n+k)$ matrix

$$A_1=\begin{pmatrix} a_{1\,1} & \cdots & a_{1\,m} & a_{1\,m+1} & \cdots & a_{1\,n} & p_{11} & \cdots & p_{1\,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ a_{n\,1} & \cdots & a_{n\,m} & a_{n\,m+1} & \cdots & a_{n\,n} & p_{n\,1} & \cdots & p_{n\,k} \\ d_{1\,1} & \cdots & d_{1\,m} & 0 & \cdots & 0 & c_{1\,1} & \cdots & c_{1\,k} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ d_{k\,1} & \cdots & d_{k\,m} & 0 & \cdots & 0 & c_{k\,1} & \cdots & c_{k\,k} \end{pmatrix}$$

and B_1 is an $(n+k)\times r$ matrix as shown below.

$$B_1=\begin{pmatrix} b_{1\,1} & \cdots & b_{1\,r} \\ \vdots & \vdots & \vdots \\ b_{n\,1} & \cdots & b_{n\,r} \\ 0 & \cdots & 0 \\ \vdots & \vdots & \vdots \\ 0 & \cdots & 0 \end{pmatrix}.$$

Suppose, also, that there is an additional condition for the system (2) (or (4)) which assumes that the states $y_1,\ldots,y_k$ remain close to the point $O(0,\cdots,0)$, $y(t_1)=y_1$ is infinitely close to zero and there is a constraint given on the system (2) (or (4)). Now suppose that the constraint is given as

$$J[\bullet]=\int_{t_0}^{\infty}\left(\sum_{i,j=1}^{k}\alpha_{ij}y_iy_j+\sum_{i,j=1}^{m}\beta_{ij}x_ix_j\right)dt. \qquad (7)$$

Thus, we can choose $x_1,\ldots,x_m$ to be control actions for the system (2) (or (4)), and hence, we can define to the following problem.

Problem Definition 2. Assume we are given the dynamics of the state space model (2) (or (4)) and the constraint (7). We need to find the control actions $\bar{x}_1^0[t],\ldots,\bar{x}_m^0[t]$ such that the system (2) (or (4)) becomes asymptotically stable and the constraint (7) reaches its minimal value.

Now, because of the assumption (5) the system (2) (or (4)) becomes fully controllable [1], hence, for any reasonable initial position $y(t_0)=y_0$ there exists unique $(x_1^0\cdots x_m^0)^T$ column vector of control actions which solve the problem 2 [2]. This means that also the states $y_1^0(t),\ldots,y_k^0(t)$ will be calculated uniquely, moreover

$$\lim_{t\to\infty}\bar{x}_i^0[t]=0, \quad (i=1,\ldots,m) \qquad (8)$$

and

$$\lim_{t\to\infty}y_i^0(t)=0, \quad (i=1,\ldots,k). \qquad (9)$$

Now that we solved the second problem, we will discuss the problem 1. So, by substituting the functions $\bar{x}_1^0[t],\ldots,\bar{x}_m^0[t]$ and $y_1^0(t),\ldots,y_k^0(t)$, which we gained by solving the problem 2, into the system (1) (or (3)), we can rewrite the system as

$$\dot{x}_i=a_{i\,m+1}x_{m+1}+\ldots+a_{i\,n}x_n+ \\ +b_{i\,1}u_1+\ldots+b_{i\,n}u_n+f_i(t) \qquad (10)$$

where $i=m+1,\ldots,n$, and

$$f_i(t)=a_{i\,1}\bar{x}_1^0[t]+\ldots+a_{i\,m}\bar{x}_m^0[t]+ \\ +p_{i\,1}y_1^0(t)+\ldots+p_{i\,k}y_k^0(t) \qquad (11)$$

where $i=1,\ldots,n$. It is obvious that first m equations from the system (10) will become algebraic equations because the functions $x_1^0[t],\ldots,x_m^0[t]$ will be already known.

According to (6) the system (1)-(2) is fully controllable, hence, it remains to calculate the control actions $u=(u_1(t)\cdots u_r(t))^T$ which solve the first problem, and that can be done by choosing some known algorithm. Thus, the problem is solved.

3. Differential Equations of Motion and the Definition of the Problem

Assume we have a cart with mass M that can move freely on a horizontal plane. Furthermore, assume that there is an inverted pendulum made of a weightless rod of length l and a small ball of mass m attached to the center of mass C of the cart end. Assume the pendulum is attached to the cart with a ball joint (Fig. 1):

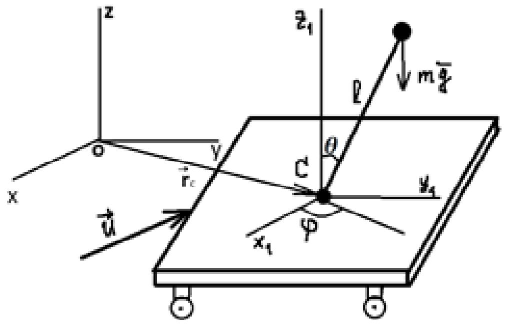

Fig. 1.

Let us introduce a fixed coordinate system Oxyz and a coordinate system $Cx_1y_1z_1$ attached to the center of mass C of the cart to study the movement of the abovementioned mechanical system. It is also assumed that the axis of $Cx_1y_1z_1$ remains parallel to corresponding axis of $Oxyz$ while the mechanical system moves and the control action $\vec{u}$ which is in the plane Oxy acts on the cart.

In that case, the coordinates of the ball in $Oxyz$ will be the following

$$x = x_C + l\sin\theta\cos\varphi, \qquad y = y_C + l\sin\theta\sin\varphi, \tag{12}$$

where $(x_C, y_C, 0)$ are the coordinates of the center of mass of the cart C in the fixed coordinate system, and (l, θ, φ) are the spherical coordinates of the ball in $Cx_1y_1z_1$.

Let us write the differential equations of the mechanical system. The kinetic energy of the system will be as follows.

$$T = \frac{1}{2}M\left(\dot{x}_C^2 + \dot{y}_C^2\right) + \frac{1}{2}m\Big(\dot{x}_C^2 + \dot{y}_C^2 + l^2\cos^2\theta\cdot\dot{\theta}^2 + l^2\sin^2\theta\cdot\dot{\varphi}^2 + 2l\dot{x}_C\dot{\theta}\cos\theta cos\varphi - 2l\dot{x}_C\dot{\varphi}\sin\theta\sin\varphi + 2l\dot{y}_C\dot{\theta}\cos\theta\sin\varphi + 2l\dot{y}_C\varphi\sin\theta\cos\varphi\Big), \tag{13}$$

As for the potential energy, we will have

$$\Pi = -mgl(1-\cos\theta): \tag{14}$$

Using the Lagrangian [7] the mechanical system we will have the below system as a linear approximation of the differential equations of the motion.

$$\begin{cases} (M+m)\ddot{x}_C + ml\ddot{\theta} = u_x, \\ (M+m)\ddot{y}_C = u_y, \\ l\ddot{\theta} + \ddot{x}_C = -g\theta. \end{cases} \tag{15}$$

where u_x and u_y are the projections of control aciton $\vec{u}$ in the coordinate system $Oxyz$. It is easy to check that the system (15) is fully controllable. Let us now define the following problem.

Problem 1.1. Given the system (15), the time interval $[t_0, t_1]$, initial position $x_C(t_0) = x_{C0}$, $y_C(t_0) = y_{C0}$, $\dot{y}_C(t_0) = \dot{y}_{C0}$, $\theta(t_0) = \theta_0$, $\dot{\theta}(t_0) = \dot{\theta}_0$ and the desired final position $x_C(t_1) = x_{C1}$, $y_C(t_1) = y_{C1}$, $\dot{y}_C(t_1) = \dot{y}_{C1}$ for the system (15). Find a control action $\vec{u}$ such that it drives the system from its given initial position to its final position in $[t_0, t_1]$ while keeping the pendulum close to its upper equilibrium point.

4. Solution of the Problem

We will solve this problem using the hybrid control method mentioned in part 1. Let us consider only the last equation of the system (15) which is

$$l\ddot{\theta} + \ddot{x}_C = -g\theta. \tag{16}$$

Let us make the notations $x_1 = \theta$, $x_2 = \dot{\theta}$, and consider $\ddot{x}_C$ as a control action. We will then have

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -k^2x_1 + u_1. \end{cases} \tag{17}$$

Here $k^2 = \frac{g}{l}$, $u_1 = -\frac{\ddot{x}_C}{l}$. It is easy to check that the system (17) is fully controllable as well.

Now let us state an optimal stabilization problem for the system (17).

Problem 2.1. Assume we are given the control system (17). It is required to find a control action $u_1^0(x_1, x_2, t)$ such that the system (17) becomes asymptotically stable when $u_1 = u_1^0(x_1, x_2, t)$ and the constraint

$$J[\cdot] = \int_0^\infty (x_2^2 + u_1^2)\,dt \tag{18}$$

reaches its minimal value.

We will solve the second problem using Lyapunov-Belman method [8]. Belman's expression will have the form

$$B[\cdot] = x_2 + \frac{\partial V}{\partial x_2}(u_1 - k^2x_1) + x_2^2 + u_1^2.$$

Here V is Lyapunov function. For the optimal control action, we know

$$\left.\frac{\partial B}{\partial u_1}\right|_{u_1 = u_1^0} = 0,$$

Thus

$$u_1^0 = -\frac{1}{2}\frac{\partial V}{\partial x_2}. \tag{19}$$

Hence, to get the optimal Lyapunov's function we will get

$$\frac{\partial V}{\partial x_2}x_2 - k^2\frac{\partial V}{\partial x_2}x_1 + x_2^2 - \left(\frac{\partial V}{\partial x_2}\right)^2 = 0. \tag{20}$$

differential equation with partial derivatives. We will look for a Lyapunov's function as a quadratic form. From (20) we will have

$$V^0(x_1, x_2) = k^2x_1^2 + x_2^2, \quad \text{thus } u_1^0 = -x_2 = -\dot{\theta}.$$

Substituting into (17) we get

$$\begin{cases} \dot{x}_1 = x_2, \\ \dot{x}_2 = -k^2x_1 - x_2. \end{cases} \tag{21}$$

Solving the system (21) for $k > 0,5$ we will have for the spherical pendulum the optimal deflection angle from the vertical position and its angular velocity as functions of time:

$$\theta^0(t) = x_1^0(t) = e^{-0,5t}\left(A\sin\sqrt{4k^2-1}\,t + \right. \\ \left. + B\cos\sqrt{4k^2-1}\,t\right), \quad (22)$$

$$\dot\theta^0(t) = x_1^0(t) = \\ = e^{-0,5t}\left(\left(-0,5A - B\sqrt{4k^2-1}\right)\sin\sqrt{4k^2-1}\,t + \right. \\ \left. + \left(A\sqrt{4k^2-1} - 0,5B\right)\cos\sqrt{4k^2-1}\,t\right) \quad (23)$$

For the case when $0 < k \leqslant 0,5$ the solutions will be exponentially decreasing functions. We will not show those solutions here.

A and B in (22) and (23) are integration constants and are calculated using the initial conditions $\theta(t_0) = \theta_0$ and $\dot\theta(t_0) = \dot\theta_0$:

$$A = \frac{2\dot\theta_0 + \theta_0}{2\sqrt{4k^2-1}}, \quad B = \theta_0. \quad (24)$$

Now let us get back to solving the Problem 1.1. According to the notation $u_1 = -\frac{\ddot x_C}{l}$ we will have

$$\ddot x_c(t) = -lu_1^0(t) = l\dot\theta^0(t) = \\ = le^{-0.5t}\left(\left(-0,5A - B\sqrt{4k^2-1}\right)\sin\sqrt{4k^2-1}\,t + \right. \\ \left. + \left(A\sqrt{4k^2-1} - 0,5B\right)\cos\sqrt{4k^2-1}\,t\right),$$

thus

$$\dot x_C(t) = l\theta^0(t) + C_1 = \\ = le^{-0,5t}(A\sin\sqrt{4k^2-1}\,t + B\cos\sqrt{4k^2-1}\,t) + C_1, \\ x_C(t) = \frac{le^{-0,5t}}{4k^2 - 0,75}\left(\left(B\sqrt{4k^2-1} - 0,5A\right)\cdot \right. \\ \cdot\sin\sqrt{4k^2-1}\,t - \left(A\sqrt{4k^2-1} + 0,5B\right)\cdot \\ \left. \cdot\cos\sqrt{4k^2-1}\,t\right) + C_1 t + C_2. \quad (25)$$

Using the initial and the desired final positions $x_C(t_0) = x_{C0}$ and $x_C(t_1) = x_{C1}$ from the system (25) we can calculate the integration constants C_1 and C_2.

Let us now calculate the u_x part of the control action. From the first equation of (15) we get

$$u_x(t) = (M+m)\ddot x_C + ml\ddot\theta = le^{-0,5t}\cdot \\ \cdot\left(\left((M+m)\left(-B\sqrt{4k^2-1} - 0,5A\right) + \right.\right. \\ \left. + m\left((1,25 - 4k^2)A + B\sqrt{4k^2-1}\right)\right)\cdot \\ \cdot\sin\sqrt{4k^2-1}\,t + \\ + \left((M+m)\left(A\sqrt{4k^2-1} - 0,5B\right) + \right. \\ \left. + m\left((1,25 - 4k^2)B - A\sqrt{4k^2-1}\right)\right)\cdot \\ \left. \cdot\cos\sqrt{4k^2-1}\,t\right) \quad (26)$$

It remains to solve only the second equation of (15) which is

$$(M+m)\ddot y_C = u_y \quad (27)$$

This equation itself is a simple control equation. We can add a cost function and define an optimal control problem for (27) which we can solve by any known method (e.g. the method of solution may be a problem of momentums). However, we will not present the problem in this paper.

5. Numerical Example

Let us introduce the below values to calculate the control actions and the trajectories as functions of time only.

$M = 10kg$, $m = 1kg$, $l = 0.5m$, $t_0 = 0s$, $t_1 = 50s$, $x_C(t_0) = 0m$, $y_C(t_0) = 0m$, $\theta(t_0) = 0.5$, $\theta(t_0) = 1s^{-1}$, $x_C(t_1) = 3m$, $y_C(t_1) = 2m$.
We will have

$$\theta^0(t) = e^{-0.5t}(0.1421\sin 8.7977t + 0.5\cos 8.7977t), \\ \dot\theta^0(t) = e^{-0.5t}(-4.4699\sin 8.7977t + 0.9922\cos 8.7977t), \\ x_c(t) = 0.0097 + 0.0598t + 0.0064e^{-0.5t}(-1.5\cos[8.7977t] + \\ + 4.3278\sin[8.7977t]), \\ \dot x_C(t) = 0.5e^{-0.5t}(0.1421\sin 8.7977t + \\ + 0.5\cos 8.7977t) + 0.0598, \\ u_x(t) = -0.5e^{-0.5t}(28.825\cos 8.7977t + \\ + 55.7317\sin 8.7977t)$$

State trajectories and graphs of control actions are constructed and shown in pictures 2-4.

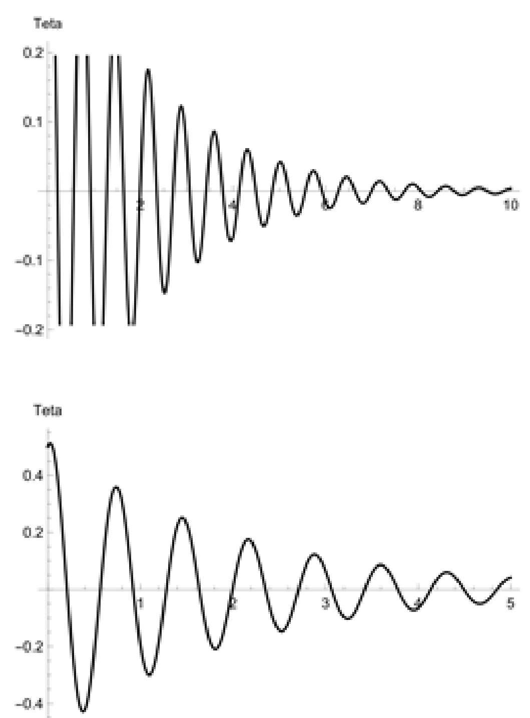

Fig. 2. The graph of the change in the $\theta^0(t)$ angle of deviation of the pendulum from the vertical as a function of time.

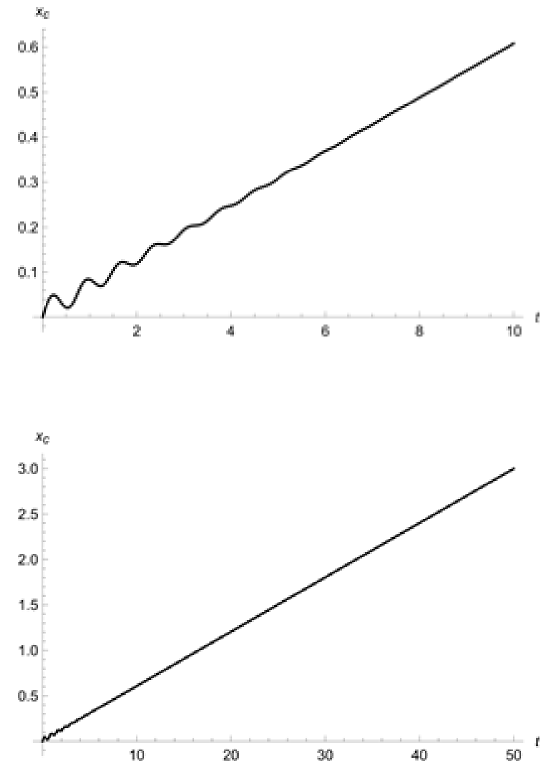

Fig. 3. The graph of the change in the $x_C(t)$ coordinate of the center of mass of the pendulum as a function of time.

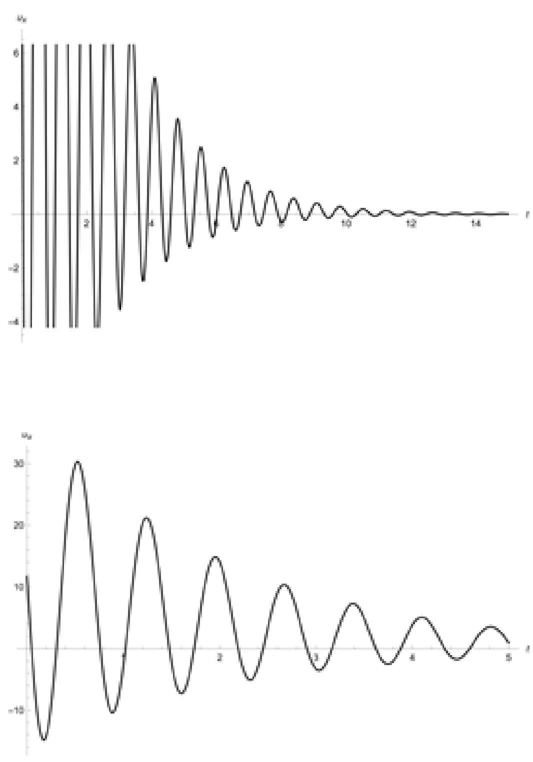

Fig. 4. Graph of control action $u_x(t)$

6. Conclusion

A hybrid control problem of a system of linear differential equations with constant coefficients is discussed in this paper. It was assumed that some of the states of the system have to satisfy some additional conditions. To ensure those conditions are satisfied, some of the states of one subsystem were chosen to be additional control actions in second subsystem. Then, an optimal stabilization problem was defined and solved for the second subsystem using Lyapunov-Bellman method. The special states which were chosen to be control actions and the corresponding optimal trajectories were acquired for the second subsystem. Afterwards, those solutions are substituted in the first subsystem and the main control problem was solved. An example of a hybrid control problem of an inverted spherical pendulum with a moving base is studied. The pendulum is chosen as a subsystem the motion of which is controlled by the moving base (the cart). Analytical representations of the states and control action are calculated and presented. The optimal trajectories and the graph of the control action are constructed for a numerical example.

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