Wada Basins Colourings and Ergodicity

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Abstract: - The stable Birkhoff curve ergodic properties in dissipative situation have been discussed. The points of every regions have been coloured with their own colour, such that their common boundary would turns out to be white at the colours adding together. Quantitative conditions for colouring regions for boundary discolouration have been obtained. The prime example of the dynamic system action with fixed B- saddle and Birkhoff curve being four regions common boundary has been constructed. There subsist the Birkhoff curves different topologic types. For every Birkhoff curves can constructed to be two or more vortex streets with periodic structure.The regions colouring turns out to be the topological classification instrument.

Keywords: -Wada basins, Wada property, indecomposable continuum (atom), composanta, rotation number, γ-density, Schnirelmann density, ergodic theorem, PostScript

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1. General Remarks

Recently authors the Birkhoff curves additive theory has been constructed [1, 2, 3]. The Birkhoff curves were researched to be invariant continua with respect to the dissipative dynamic systems $\psi\colon \mathbb{N} \to \operatorname{Diff}(\mathbb{E}^2)$ acting on Euclidean plane defined by the formula

$$P(x,y) + iQ(x,y) \overset{def}{\mapsto} x + iy \tag{1}$$

at iterations. Also the authors had constructed the Birkhoff curves at presence of prime or complex equilibrium point [4].

Inter alia there subsists invariant set Λ (i.e. $\Lambda \overset{def}{=} \psi_k(\Lambda)$ for all $k \in \mathbb{Z}$) on the plane, such that a homomorphism action

$$\psi\colon \mathbb{Z} \to \operatorname{Diff}(\Lambda)$$

has been defined by the formula (1).

The Birkhoff curve separates the plane and one turns out to be indecomposable continuum (atom) (even if Birkhoff curve separates the plane on two regions). Then the invariant set Λ with respect to dissipative action ψ is constituted to be plane connected compact.

The colouring invariant regions problem has been solved author in [5, 6]. Also this problem solved in [9, 10, 11]. However the colouring problem every time had only empiric solution.

Suppose Υ be a Birkhoff curve being two or more invariant regions common boundary on the plane and $\Upsilon \subset \Lambda$. If action ψ turns out to be dissipative then addition $\mathbb{E}^2 \setminus \Lambda$ be only semi-invariant set.

Now suppose that the Birkhoff curve Υ is either the boundary of Λ, or Υ separates Λ into ν invariant regions G_i, $i \in \overline{1,\nu}$, being their common boundary. Then the following equalities

$$\Upsilon \overset{def}{=} \operatorname{Fr} G_1 = \ldots = \operatorname{Fr} G_\nu = \operatorname{Fr}(\mathbb{E}^2 \setminus \Lambda) \overset{def}{=} \operatorname{Fr} G_0$$

are faithful.

Remark 1 *The Birkhoff curve* Υ *is constituted to be* (*atom*) *and therefore one consists of only tops of 1-umbrellas, because the oscillations in every point of atom is equal to its diameter* (*see* [1, 13] and e. g. [12], *vol. II or* [14]).

Therefore all tops of 1-umbrellas containing in Birkhoff curve turn out to be irreducible points. It is clear that the Birkhoff curve is not a manifold, even if it turns out to be two regions common boundary.

The *accessible point* from region G_i, $i \in \overline{1,\nu}$ is said to be point $\overset{\circ}{q}_i \in \Upsilon$, such that there exists arc γ_i with end being $\overset{\circ}{q}_i$ and $\gamma_i \setminus \{\overset{\circ}{q}_i\} \in G_i$. The *Wada basins* $\overset{\circ}{G}_i$ are called to be invariant regions G_i and set of all accessible points from G_i union for all $i \in \overline{1,\nu}$, if $\nu \geqslant 2$ (see also e. g. [3]). The *Wada ocean* $\overset{\circ}{G}_0$ is said to be invariant region G_0 and set of all accessible points from G_0 union.

In this connotation, the following statements Υ *is the Birkhoff curve* and $\overset{\circ}{G}_i \cap \overset{\circ}{G}_j = \varnothing$ *for all* $i, j \in \overline{0,\nu}$, $\nu \geqslant 1$ *and* $i \neq j$ are equivalent (see [15]). However there are no reasons to assume that Υ does not contain inaccessible points from any of the invariant regions.

For simplicity of further narration suppose that Birkhoff curve turns out to be ω-limit continuum for dissipative action ψ. Then there subsist ν fixed or periodic unstable antisaddles contained into invariant regions with compact closure. This means that the points of every trajectory accumulate close enough to Υ, with the exception, perhaps, of the fixed or periodic unstable antisaddles. Moreover every trajectory close enough to Υ everywhere dense, with the exception, perhaps, of the fixed or periodic points (for this occasion, see [1]). This dynamic systems property Poincaré has been called to be *ergodic* (see e. g. [7]). It should be noted that the action ψ is dissipative, but the invariant regions are incompressible.

Now let us colouring the Wada ocean and every Wada basin in its own colour defined, for instance, by the triple of positive numbers not exceeding one (r_i, g_i, b_i), $i \in \overline{0,\nu}$ so, that

$$\sum_{0 \leqslant i \leqslant \nu} (r_i, \mathrm{g}_i, b_i) \equiv (1,1,1). \qquad (2)$$

Then what should be the ¡¡colouring density¿¿ of every invariant region (or Wada ocean and basins) for their common border turns out to be white; or that too it has been defined by triple of ones $(1,1,1)$ in the neighborhood of every the boundary point?

2. Wada Basins Colouring

A successful colouring has been obtained at the dynamic system constructing with two invariant centrally symmetric Wada basins [6]. However, this colouring turned out intuitive or empirical.

First us define what is the *colouring density.* Suppose $u_1, u_2, \ldots, u_\nu$ are unstable antisaddles with their neighbourhoods

$$U(u_1), U(u_2), \ldots, U(u_\nu),$$

such that $U(u_i) \subset G_i$ for all $i \in \overline{1,\nu}$. In addition, let us choose some point u_0 in G_0 with neighbourhood $U(u_0)$, such that $U(u_0) \subset G_0$.

For definiteness one can suppose that all the neighbourhoods $U(u_i)$ appear identical open small squares. Then the coloured points number $\#U(u_i)$ is called to be the *region G_i colouring density.*

So, coloured the open small squares of the colouring densities

$$\#U(u_0), \#U(u_1), \ldots, \#U(u_\nu)$$

have been determined.

Due to the fact that continuum Υ appears stable boundary there are true the following

Proposition 1 *For any neighbourhood* $U(\xi)$ *of the point* $\xi \in \Upsilon$ *there exists number* $k \in \mathbb{N}$, *such that for the forms of the map*

$$\psi_k(U(u_0)), \psi_k(U(u_1)), \ldots, \psi_k(U(u_\nu))$$

the intersections $\psi_k(U(u_i)) \cap U(\xi)$ *are non-empty for all* $i \in \overline{0,\nu}$.

Corollary 1 *For a sufficiently large* $k \in \mathbb{N}$ *the inequality* $\#(\psi_k(U(u_i)) \cap U(\xi)) \geqslant 1$ *is faithful.*

Suppose G_i is appeared to be invariant region with respect to the action ψ containing neighbourhood $U(u_i)$ of periodic point u_i. Now assume $U(u_i)$ contains either $\#U(u_i)$ colouring points or the only colouring point $c_i \neq u_i$, if $i \in \overline{1,\nu}$. In these ways, one can start colouring the Wada basins. However for G_0 one can colourize point

$u_0 = c_0$ and neighbourhood $U(c_0)$ containing either $\#U(c_0)$ colouring points or the only colouring point c_0. In these ways, one can start colouring the Wada ocean.

In the sequel, the basins and ocean can be colored in the following two ways, either the forms of the neighbourhoods $U(u_i)$ at iterations

$$U(u_i),\, \psi_1(U(u_i)),\, \dots,\, \psi_K(U(u_i)),\, \dots$$

with $\#U(u_i)$ colouring points colourize the invariant regions or semi-trajectories $\mathcal{O}_+(c_i)$ colourize the invariant regions for all $i \in \overline{0,\nu}$. However the colours regions satisfy to condition (2) and the colouring points numbers ratios or the lengths of colourized semi-trajectories ratios must appeared to be such that the boundary continuum (Birkhoff curve) turns out to be white.

Remark 2 *The condition* (2) *turns out to be scalar. Indeed one can apply ¡¡grayscales¿¿ S_i for colouring as follows $\sum\limits_{0\leqslant i\leqslant\nu} S_i \equiv 1$. It means that for every region colouring by the only* (*scalar*) *parameter is determined.*

It is quite natural to consider numerical topological or metric invariants of the region, for instance the *rotation number* or *γ-density* [2], as a colouring parameters.

3. Densities and Colourings

The additive rotation number theory has been constructed in application to the Birkhoff curve (see [1]). The theory is interesting but one is trivial applied to Jordan curve.

Birkhoff curves turn out to be nonwandering continua Υ with respect to the dissipative dynamic systems ψ acting on the plane, such that Υ separates the plane. Its appear in different dynamic situations.

For every invariant region G_l, $l \in \overline{0,\nu}$ and for Wada basin $\overset{\circ}{G}_i$ the iterations numbers sequence $\mathfrak{K}_l$: $k_1 = 1, k_2, k_3, \dots$ has been defined. *Rotation number* for $\overset{\circ}{G}_i$ being *Schnirelmann density* for $\mathfrak{K}_l$ is defined to be following formula

$$\sigma\mathfrak{K}_l \overset{def}{=} \inf_{K\in\mathbb{N}} \frac{\#(\{\overline{1,K}\} \cap \mathfrak{K}_l)}{K} \qquad (3)$$

(compare with the definition e. g. from [8]). Such a way the set

$$\{\sigma\mathfrak{K}_0,\, \sigma\mathfrak{K}_1,\, \dots,\, \sigma\mathfrak{K}_l,\, \dots,\, \sigma\mathfrak{K}_\nu\}$$

turns out to be topological invariant for boundary Υ. The Schnirelmann densities for $\mathfrak{K}_l$ corresponding to $\overset{\circ}{G}_i$ turn out to be irrational numbers. It means that that accessible point trajectory is everywhere dense on Υ (the theory details, see in [1]).

Proposition 2 *If rotation numbers of basins are equal then the colouring rates of their common boundary are equal.*

PROOF. Indeed, if the basins $\overset{\circ}{G}_i$ and $\overset{\circ}{G}_j$, $i, j \in \overline{0,\nu}$, such that their rotation numbers are equal, i. e. $\sigma\mathfrak{K}_i = \sigma\mathfrak{K}_j$ for $i \neq j$. Then there exists number $\operatorname{ord}\mathfrak{K}_i = \operatorname{ord}\mathfrak{K}_j \overset{def}{=} \eta$, such that

$$\sigma(\eta \oplus \mathfrak{K}_i) = \sigma(\eta \oplus \mathfrak{K}_j) \equiv 1,$$

or that too

$$\eta \oplus \mathfrak{K}_i = \eta \oplus \mathfrak{K}_j \equiv \mathbb{N}.$$

It means that colouring rate of region G_i is equal to colouring rate of region G_j □

Theorem 1 *The difference in the rotation numbers of regions* (*basins*) *means the colouring rate difference of their common boundary.*

PROOF. Suppose $\overset{\circ}{G}_i$ and $\overset{\circ}{G}_j$, $i, j \in \overline{0,\nu}$, such that their rotation numbers are different, i. e. for instance $\sigma\mathfrak{K}_i > \sigma\mathfrak{K}_j$ for $i \neq j$. Then there subsist two numbers $\operatorname{ord}\mathfrak{K}_i \geqslant \operatorname{ord}\mathfrak{K}_j$ (called the *order sequence*), such that

$$\sigma(\operatorname{ord}\mathfrak{K}_i \oplus \mathfrak{K}_i) = \sigma(\operatorname{ord}\mathfrak{K}_j \oplus \mathfrak{K}_j) \equiv 1,$$

or that too

$$\operatorname{ord}\mathfrak{K}_i \oplus \mathfrak{K}_i = \operatorname{ord}\mathfrak{K}_j \oplus \mathfrak{K}_j \equiv \mathbb{N}.$$

It means that colouring rate of region G_j does not exceed of colouring rate of region G_i. In the event that $\operatorname{ord}\mathfrak{K}_i = \operatorname{ord}\mathfrak{K}_j$ let us cross it out, for instance, even elements from the sequences $\mathfrak{K}_i$ and $\mathfrak{K}_j$. Thus new sequences $\mathfrak{K}_i'$ and $\mathfrak{K}_j'$ consisting of odd elements of the sequences $\mathfrak{K}_i$ and $\mathfrak{K}_j$ are obtained. Then $\sigma\mathfrak{K}'_i > \sigma\mathfrak{K}'_j$ and there subsist two numbers $\operatorname{ord}\mathfrak{K}'_i \geqslant \operatorname{ord}\mathfrak{K}'_j$, such that

$$\sigma(\operatorname{ord}\mathfrak{K}'_i \oplus \mathfrak{K}'_i) = \sigma(\operatorname{ord}\mathfrak{K}'_j \oplus \mathfrak{K}'_j) \equiv 1.$$

Next, the process of crossing out even elements of sequences is repeated until for new two sequences $\sigma\mathfrak{K}''_i > \sigma\mathfrak{K}''_j$ their orders will satisfy to the inequality $\operatorname{ord}\mathfrak{K}_i > \operatorname{ord}\mathfrak{K}_j$ □

Remark 3 *Inter alia crossing out scheme can be anything, if only it was the same for both sequences $\mathfrak{K}_i$ and $\mathfrak{K}_j$.*

Moreover if Υ turns out to be Birkhoff curve then sequence $\mathfrak{K}_l$ contains zero Schnirelmann density *additive basis* $\mathfrak{B}_l$ for all $\mathring{G}_l$ and $l \in \overline{0,\nu}$. In order to compare sequences zero Schnirelmann density the γ-*density* has been applied defined by the formula

$$\gamma\,\mathfrak{B}_l \overset{def}{=\!=} \inf_{K\in\mathbb{N}} \frac{\log(1 + \#(\overline{1,K} \cap \mathfrak{B}_l))}{\log(K+1)} \qquad (4)$$

The γ-density theory in detail has been narrated in [2, 3].

Due to the sequences $\mathfrak{B}_0, \mathfrak{B}_1, \ldots, \mathfrak{B}_\nu$ turn out to be additive bases then for every $l \in \overline{0,\nu}$ there exist numbers

$$\operatorname{ord}\mathfrak{B}_0, \operatorname{ord}\mathfrak{B}_1, \ldots, \operatorname{ord}\mathfrak{B}_\nu,$$

such that $\sigma(\operatorname{ord}\mathfrak{B}_l \oplus \mathfrak{B}_l) \equiv 1$.

Proposition 3 *If γ-densities defined for regions (basins) are equal then the colouring rates of their common boundary are equal.*

PROOF exactly repeats the proof of the proposition (2) □

Theorem 2 *The difference in γ-densities defined for regions (basins) means the colouring rate difference of their common boundary.*

PROOF. Suppose $\mathring{G}_i$ and $\mathring{G}_j$, $i, j \in \overline{0,\nu}$, such that their γ-densities are different, i. e. for instance $\gamma\mathfrak{B}_i > \gamma\mathfrak{B}_j$ for $i \neq j$. Then there exist two numbers $\operatorname{ord}\mathfrak{B}_i \geqslant \operatorname{ord}\mathfrak{B}_j$, such that

$$\sigma(\operatorname{ord}\mathfrak{B}_i \oplus \mathfrak{B}_i) = \sigma(\operatorname{ord}\mathfrak{B}_j \oplus \mathfrak{B}_j) \equiv 1,$$

or that too

$$\operatorname{ord}\mathfrak{B}_i \oplus \mathfrak{B}_i = \operatorname{ord}\mathfrak{B}_j \oplus \mathfrak{B}_j \equiv \mathbb{N}.$$

It means that colouring rate of region G_j does not exceed of colouring rate of region G_i same as for rotation numbers. In the event that $\operatorname{ord}\mathfrak{B}_i = \operatorname{ord}\mathfrak{B}_j$ at the crossing out even elements from the sequences $\mathfrak{K}_i$ and $\mathfrak{K}_j$ the sequences $\mathfrak{K}'_i$ and $\mathfrak{K}'_j$, containing $\mathfrak{B}'_i \subset \mathfrak{B}_i$ and $\mathfrak{B}'_j \subset \mathfrak{B}_J$ respectively, are obtained. Thus new bases $\mathfrak{B}'_i$ and $\mathfrak{B}'_j$ consisting of odd elements of the sequences $\mathfrak{K}_i$ and $\mathfrak{K}_j$ have been obtained. Therefore $\gamma\,\mathfrak{B}'_i > \gamma\,\mathfrak{B}'_j$ and there subsist two numbers $\operatorname{ord}\mathfrak{B}'_i \geqslant \operatorname{ord}\mathfrak{B}'_j$, such that

$$\sigma(\operatorname{ord}\mathfrak{B}'_i \oplus \mathfrak{B}'_i) = \sigma(\operatorname{ord}\mathfrak{B}'_j \oplus \mathfrak{B}'_j) \equiv 1.$$

Next, the process of crossing out even elements of sequences is repeated until for new two sequences $\sigma\mathfrak{K}''_i > \sigma\mathfrak{K}''_j$ their orders will satisfy the inequality $\operatorname{ord}\mathfrak{B}''_i > \operatorname{ord}\mathfrak{B}''_j$ for both bases $\mathfrak{B}''_i \subset \mathfrak{K}''_i$ and $\mathfrak{B}''_j \subset \mathfrak{K}''_j$ respectively □

Thus, the colouring rates are only provided by the existence of additive bases zero Schnirelmann density $\mathfrak{B}_l$ with γ-densities values for all different invariant regions. However the colouring rates are determined by corresponding to these regions the rotation numbers.

4. Invariant Regions Colouring

Let us place

$$\#U(u_0), \#U(u_1), \ldots, \#U(u_l), \ldots, \#U(u_\nu)$$

colouring points contained in $U(u_l) \subset G_l$ for every invariant region respectively, and let us build for every colouring point ζ from neighbourhood $U(u_l)$ the sequence $\mathfrak{K}_l(\zeta)$ on the following scheme described in [1]:

1° suppose that an arbitrary ray B_l is selected with its beginning at the point u_l, so that one sets the direction angle φ; then for every point the direction with respect to the point u_l being equal $\arg\xi$ is defined;

2° a forms of all colouring points $\mathcal{K}_l$ contained in $U(u_l) \subset G_l$ is separated on the finite disjoint classes number

$$\mathcal{K}_l^{(1)}, \mathcal{K}_l^{(2)}, \ldots, \mathcal{K}_l^{(\eta)} \qquad (5)$$

as follows:

(1) $\zeta \in \mathcal{K}_l^{(1)}$ if $\arg(\psi_1(\zeta)) > \arg\xi$;

(2) $\zeta \in \mathcal{K}_l^{(2)}$ if $\xi \in \mathcal{K}_l \setminus \mathcal{K}_l^{(1)}$ and $\arg(\psi_2(\zeta))) > \arg \zeta$;

(3) $\zeta \in \mathcal{K}_l^{(3)}$ if $\zeta \in \mathcal{K}_l \setminus \mathcal{K}_l^{(1)} \setminus \mathcal{K}_l^{(2)}$ and $\arg(\psi_3(\xi))) > \arg \zeta$;

$\dots \dots \dots \dots \dots \dots \dots \dots \dots \dots \dots$

(η) $\zeta \in \mathcal{K}_l^{(\eta)}$ if $\zeta \in \mathcal{K}_l \setminus \mathcal{K}_l^{(1)} \setminus \dots \setminus \mathcal{K}_l^{(\eta-1)}$ and $\arg(\psi_\eta \zeta))) > \arg \zeta$;

disjoint classes number turns to be finite due to $\sigma\mathfrak{K}_l > 0$, thus $\eta = \operatorname{ord} \mathfrak{K}_l$ is finite for any point $\xi \in U(u_l)$, and then

$$\mathcal{K}_l \stackrel{def}{=} \mathcal{K}_l^{(1)} \cup \mathcal{K}_l^{(2)} \cup \dots \cup \mathcal{K}_l^{(\eta)}$$

and along with that

$$\#U(u_l) \stackrel{def}{=} \#\mathcal{K}_l^{(1)} + \#\mathcal{K}_l^{(2)} + \dots + \#\mathcal{K}_l^{(\eta)};$$

3° the colouring points from the set

$$\mathcal{U}_K(u_l) \stackrel{def}{=} U(u_l) \cup \psi_1(U(u_l)) \cup \dots \cup \psi_K(U(u_l))$$

is separated on the finite disjoint classes number (5) as well as $U(u_l)$ and full invariant region colouring has been defined by the number $\#\mathcal{U}_K(u_l)$.

Suppose $\xi_l \in \Upsilon$ be a accessible point from the region G_l and $\mathcal{O}(\xi_l) \subset \Upsilon$ be an its trajectory. All trajectory points $\psi_k(\xi_l)$ are supplied with a self neighbourhoods $U_\varepsilon(\psi_k(\xi_l))$ of the same diameter d_ε for all $k \in \mathbb{Z}$. Due to the compactness Υ there subsists number $N(\varepsilon)$, such that

$$\Upsilon \subset \bigcup_{\mu \leqslant N(\varepsilon)} U_\varepsilon(\psi_{k_\mu}(\xi_l)) \stackrel{def}{=} \Omega_\varepsilon(\Upsilon).$$

Then for any $\varepsilon > 0$ there exists $k > K$, such that $\Omega_\varepsilon(\Upsilon) \cap \mathcal{U}_K(u_l) \neq \emptyset$, or that too

$$\#(\Omega_\varepsilon(\Upsilon) \cap \mathcal{U}_K(u_l)) > 0.$$

Indeed for any colouring point a form of the map (1) at iterations due to the boundary Υ stability turns out to be sufficiently close to Υ. Inter alia index l for the coverage $\Omega_\varepsilon(\Upsilon)$ building can choose any from $\overline{0,\nu}$. Then any neighbourhood $U_\varepsilon(\psi_{k_\mu}(\xi_l))$ contains all colours points. In addition the coverage $\Omega_\varepsilon(\Upsilon)$ turns out to be minimal in the sense that

$$\Omega_\varepsilon(\Upsilon) \setminus U_\varepsilon(\psi_{k_\mu}(\xi_l))$$

is not connected for all $\mu \in \overline{1, N(\varepsilon)}$.

All colours points full number in any neighbourhood $U_\varepsilon(\psi_{k_\mu}(\xi_l))$ by the number every colouring points summation has been determined

$$\#(U_\varepsilon(\psi_{k_\mu}(\xi_l)) \cap G_0) + \dots + \#(U_\varepsilon(\psi_{k_\mu}(\xi_l)) \cap G_\nu).$$

Theorem 3 *Suppose $\mathcal{K}_i \subset \mathcal{U}_K(u_i) \subset G_i$ and $\mathcal{K}_j \subset \mathcal{U}_K(u_j) \subset G_j$, $i, j \in \overline{0,\nu}$, are colouring points from the different regions. Then*

$$\#U(u_i) \colon \#U(u_j) = \sigma\mathfrak{K}_i \colon \sigma\mathfrak{K}_j$$

for all $\varepsilon > 0$ and sufficiently large K if

$$\#(U_\varepsilon(\psi_{k_\mu}(\xi_l)) \cap \mathcal{K}_i) \colon \#(U_\varepsilon(\psi_{k_\mu}(\xi_l)) \cap \mathcal{K}_j) = 1$$

for every $\mu \in \overline{1, N(\varepsilon)}$.

An explanation of the term ¡¡for sufficiently large K¿¿ here and further will appear in the course of the future narrative (see next section).

PROOF. If $\sigma\mathfrak{K}_i = \sigma\mathfrak{K}_j$ then the statement is trivial. Therefore it is quite natural to assume $\sigma\mathfrak{K}_i \neq \sigma\mathfrak{K}_j$ for some $i, j \in \overline{0,\nu}$ and one can suppose $\sigma\mathfrak{K}_i > \sigma\mathfrak{K}_j$. Then for every K the inequality $\#\mathcal{K}_i^{(1)} > \#\mathcal{K}_j^{(1)}$ for $\mathcal{U}_K(u_i)$ and $\mathcal{U}_K(u_j)$ is faithful in the event that $\#U(u_i) = \#U(u_j)$. Thus

$$\#(\mathcal{K}_i^{(1)} \cup \mathcal{K}_i^{(2)}) > \#(\mathcal{K}_j^{(1)} \cup \mathcal{K}_j^{(2)})$$

for $\mathcal{U}_K(u_i)$ and $\mathcal{U}_K(u_j)$, because

$$\mathcal{K}_l^{(1)} \cap \mathcal{K}_l^{(2)} = \emptyset$$

for all $l \in \overline{0,\nu}$, due to

$$\sigma(\mathfrak{K}_i \oplus \mathfrak{K}_i) > \sigma(\mathfrak{K}_j \oplus \mathfrak{K}_j).$$

Now let us increase the colouring points number in neighbourhood $U(u_j)$ as follows, so that the colouring points number of class $\mathcal{K}_j^{(1)}$ has been increased to $\overline{\mathcal{K}_j^{(1)}}$ so that

$$\#\mathcal{K}_i^{(1)} = \#\overline{\mathcal{K}_j^{(1)}} = \alpha \cdot \#\mathcal{K}_j^{(1)},$$

thereby lengthening the sequence of indices of the coloured points in the region G_j. Then for sufficiently large K, the intersections $\Omega_\varepsilon(\Upsilon) \cap \mathcal{K}_i^{(1)}$ and $\Omega_\varepsilon(\Upsilon) \cap \overline{\mathcal{K}_j^{(1)}}$ possess same density for any $\varepsilon > 0$, i. e.

$$\#(\Omega_\varepsilon(\Upsilon) \cap \mathcal{K}_i^{(1)}) = \#(\Omega_\varepsilon(\Upsilon) \cap \overline{\mathcal{K}_j^{(1)}}),$$

so that

$$\#(\Omega_\varepsilon(\Upsilon) \cap \mathcal{K}_i^{(1)}) = \alpha \cdot \#(\Omega_\varepsilon(\Upsilon) \cap \mathcal{K}_j^{(1)}).$$

It is means that if $\alpha \stackrel{def}{=} \sigma\mathfrak{K}_j : \sigma\mathfrak{K}_i$ then the colouring point sets possess the same density for sufficiently large K

$$\sigma\mathfrak{K}_j \cdot \#(\Omega_\varepsilon(\Upsilon) \cap \mathcal{K}_i^{(1)}) = \sigma\mathfrak{K}_i \cdot \#(\Omega_\varepsilon(\Upsilon) \cap \mathcal{K}_j^{(1)}).$$

on following condition

$$\#(U_\varepsilon(\psi_{k_\mu}(\xi_l)) \cap \mathcal{K}_i) : \#(U_\varepsilon(\psi_{k_\mu}(\xi_l)) \cap \mathcal{K}_j) = 1$$

is faithful for all $\varepsilon > 0$, if $\#U(u_i) : \#U(u_j) = \sigma\mathfrak{K}_i : \sigma\mathfrak{K}_j$ □

Suppose for every region its colour has been defined in PostScript codes, for instance, in case of two Wada basins and Wada ocean ($\nu = 2$) as follows

$$\begin{array}{rl} & 0\ 1\ 0 \ \ \text{setrgbcolor} \\ & 1\ 0\ 0 \ \ \text{setrgbcolor} \\ & 0\ 1\ 0 \ \ \text{setrgbcolor} \\ \hline \approx & 1\ 1\ 1 \end{array}, \tag{6}$$

or that too, the boundary is coloured to be white colour. The formula (6) is faithful if $N_0 : N_1 : N_2 \approx \sigma\mathfrak{K}_0 : \sigma\mathfrak{K}_1 : \sigma\mathfrak{K}_2$ due to ¡¡different density¿¿ every dense trajectories on the boundary.

Now suppose the positive semi-trajectories

$$\mathcal{O}_+(\zeta_l), \quad l \in \overline{0, \nu}.$$

are colouring in self colours. Therefore all points of $\mathcal{O}_+(\zeta_l)$ form finite disjoint classes number as well as in 2°, where $\mathcal{K}_l \stackrel{def}{=} \mathcal{O}_+(\zeta_l)$ and K_l be a number of consecutive colouring points of $\mathcal{O}_+(\zeta_l)$ (notation $K_l \stackrel{def}{=} \#\mathcal{O}_+(\zeta_l)$). Then

$$\#\mathcal{O}_+(\zeta_l) \stackrel{def}{=} \#\mathcal{K}_l^{(1)} + \#\mathcal{K}_l^{(2)} + \ldots + \#\mathcal{K}_l^{(\eta)}$$

for any $\varepsilon > 0$ there exists $k > K_l$, such that $\Omega_\varepsilon(\Upsilon) \cap \mathcal{O}_+(\zeta_l) \neq \varnothing$, or that too

$$\#(\Omega_\varepsilon(\Upsilon) \cap \mathcal{O}_+(\zeta_l)) > 0.$$

All colouring points full number in any neighbourhood $U(\psi_{k_\mu}(\xi_l))$ from the coverage $\Omega_\varepsilon(\Upsilon)$ by the number every colouring points summation has been determined

$$\sum_{0 \leqslant l \leqslant \nu} \#(U(\psi_k(\xi_l)) \cap \mathcal{O}_+(\zeta_l)).$$

Theorem 4 *Suppose*

$$\mathcal{K}_i \subset \mathcal{O}_+(\zeta_i) \quad and \quad \mathcal{K}_j \subset \mathcal{O}_+(\zeta_j),$$

$i, j \in \overline{0, \nu}$, turn out to be colouring points from the different regions, such that $\mathcal{O}_+(\zeta_i)$ and $\mathcal{O}_+(\zeta_j)$ are any two trajectories belonging to invariant regions G_i and G_j respectively. Then

$$\#\mathcal{O}_+(\zeta_i) : \#\mathcal{O}_+(\zeta_j) = \sigma\mathfrak{K}_i : \sigma\mathfrak{K}_j$$

for all $\varepsilon > 0$ for sufficiently long $\mathcal{O}_+(\zeta_i)$ and $\mathcal{O}_+(\zeta_j)$ if

$$\frac{\#(U(\psi_{k_\mu}(\xi_l)) \cap \mathcal{O}_+(\zeta_i))}{\#(U(\psi_{k_\mu}(\xi_l)) \cap \mathcal{O}_+(\zeta_j))} = 1$$

for every $\mu \in \overline{1, N(\varepsilon)}$.

PROOF almost word for word repeats the previous theorem proof, but there are some nuances. If $\sigma\mathfrak{K}_i = \sigma\mathfrak{K}_j$ then the statement is trivial. Then one can suppose $\sigma\mathfrak{K}_i > \sigma\mathfrak{K}_j$ for some $i, j \in \overline{0, \nu}$. Therefore for every K the inequality $\#\mathcal{K}_i^{(1)} > \#\mathcal{K}_j^{(1)}$ for $\mathcal{O}_+(\zeta_i)$ and $\mathcal{O}_+(\zeta_j)$ is faithful in the event that $\#\mathcal{O}_+(\zeta_i) = \#\mathcal{O}_+(\zeta_j)$. Thus

$$\#(\mathcal{K}_i^{(1)} \cup \mathcal{K}_i^{(2)}) > \#(\mathcal{K}_j^{(1)} \cup \mathcal{K}_j^{(2)})$$

for $\mathcal{O}_+(\zeta_i)$ and $\mathcal{O}_+(\zeta_j)$, because

$$\mathcal{K}_l^{(1)} \cap \mathcal{K}_l^{(2)} = \varnothing$$

for all $l \in \overline{0, \nu}$, due to

$$\sigma(\mathfrak{K}_i \oplus \mathfrak{K}_i) > \sigma(\mathfrak{K}_j \oplus \mathfrak{K}_j).$$

Let us increase the colouring points number in trajectory $\mathcal{O}_+(\zeta_j)$ increasing the length of set the colouring points, so that class $\mathcal{K}_j^{(1)}$ has been increased to $\overline{\mathcal{K}_j^{(1)}}$ so, that

$$\#\mathcal{K}_i^{(1)} = \#\overline{\mathcal{K}_j^{(1)}} = \alpha \cdot \#\mathcal{K}_j^{(1)},$$

thereby lengthening the sequence of indices of the coloured points in the region G_j. Then exactly repeating the proof of the previous theorem one complete the proof □

Remark 4 *The Schnirelmann densities, or that too, rotation numbers for two differ Wada basins $\mathring{G}_i$ and $\mathring{G}_j$ defined by the formula (3) clearly indicate on the Birrkhoff curve ergodic properties.*

Indeed, densities for two colouring point sets of the semi-trajectories $\mathcal{O}_+(\zeta_i)$ and $\mathcal{O}_+(\zeta_j)$ different length defined by the expressions

$$\inf_{K\leqslant K_i}\frac{\#(\overline{1,K}\cap\mathfrak{K}_i)}{K}\quad\text{and}\quad\inf_{K\leqslant K_j}\frac{\#(\overline{1,K}\cap\mathfrak{K}_j)}{K}$$

and their equality for $\sigma\mathfrak{K}_i > \sigma\mathfrak{K}_j$. Then subsists sufficiently large $K_i = \inf\{K_i, K_j\}$, such that

$$\inf_{K\leqslant K_i}\frac{\#(\overline{1,K}\cap\mathfrak{K}_i)}{K}=\alpha\cdot\inf_{K\leqslant K_i}\frac{\#(\overline{1,K}\cap\mathfrak{K}_j)}{K},$$

or that too, $\alpha = \sigma\mathfrak{K}_i : \sigma\mathfrak{K}_j$.

On other hand, let us place only point different from fixed point in every invariant region by colouring the trajectories points $\mathcal{O}(x_0)$, $\mathcal{O}(x_1)$ and $\mathcal{O}(x_2)$. Then the formula (6) is faithful if $k^0 : k^1 : k^2 \approx \sigma\mathfrak{K}_0 : \sigma\mathfrak{K}_1 : \sigma\mathfrak{K}_2$ for the same reason.

Now let us formulate ¡¡synthetic condition¿¿, exactly the condition connecting of the colouring points number and the colouring semi-trajectories length. In order to formulate the condition, one suppose $\#\mathcal{O}_+(\zeta_l)$ is the colouring semi-trajectories length and $\#\zeta_l$ is the number of colouring points, such that for any pair colouring points $\#\zeta_l$ and $\#\zeta'_l$ the intersection their semi-trajectories

$$\mathcal{O}_+(\zeta_l)\cap\mathcal{O}_+(\zeta'_l)$$

turn out to be empty. Then full colouring points set has been defined by the formula

$$\mathcal{K}_l\overset{def}{=\!=}\bigcup_{\mu\leqslant\#\zeta_l}\mathcal{O}_+(\zeta_l^\mu)\tag{7}$$

forming finite disjoint classes number as well as in 2°. Therefore full colouring points number has been defined by the formula

$$\#\mathcal{K}_l\overset{def}{=\!=}\#\mathcal{O}_+(\zeta_l)\cdot\#\zeta_l,$$

so that $\Omega_\varepsilon(\Upsilon)\cap\mathcal{K}_l\neq\varnothing$, or that too

$$\#(\Omega_\varepsilon(\Upsilon)\cap\mathcal{K}_l)>0.$$

Theorem 5 *Suppose $\mathcal{K}_i\subset G_i$ and $\mathcal{K}_j\subset G_j$, $i,j\in\overline{0,\nu}$ (defined by equality (7)), are colouring points from the different regions. Then*

$$\frac{\#\mathcal{O}_+(\zeta_i)\cdot\#\zeta_i}{\#\mathcal{O}_+(\zeta_j)\cdot\#\zeta_j}=\sigma\mathfrak{K}_i:\sigma\mathfrak{K}_j\tag{8}$$

for all $\varepsilon > 0$ for sufficiently long $\mathcal{O}_+(\zeta_i)$ and $\mathcal{O}_+(\zeta_j)$, and sufficiently large K_i and K_j if

$$\#(U(\psi_{k_\mu}(\xi_l))\cap\mathcal{K}_i):\#(U(\psi_{k_\mu}(\xi_l))\cap\mathcal{K}_j)=1$$

for every $\mu\in\overline{1,N(\varepsilon)}$.

This statement turns out to be synthetic. Indeed, supposing

$$\#\mathcal{O}_+(\zeta_i)=\#\mathcal{O}_+(\zeta_j)\quad\text{or}\quad\#\zeta_i=\#\zeta_j$$

one comes to the conditions of previous theorems.

PROOF. In accordance with the remark 4 and theorem 3 due to combination theorems 3 and 4 the formula (8) is obtained □

Let us consider the following frequently occurring dynamic situation, such that point p_0 is the fixed unstable antisaddle and everyone else unstable antisaddles p_l, $l\in\overline{1,\nu}$ are ν-periodic points.

Proposition 4 $\det\begin{pmatrix}\#\mathcal{O}_+(\zeta_i) & \#\mathcal{O}_+(\zeta_j)\\ \#\zeta_j & \#\zeta_i\end{pmatrix}=1$

supposing $\#\zeta_j\neq\#\zeta_i$ or $\#\mathcal{O}_+(\zeta_i)\neq\#\mathcal{O}_+(\zeta_j)$ for all $i,j\in\overline{1,\nu}$.

PROOF. This proposition turns out to be the theorem 5 direct corollary. Indeed, if unstable antisaddles p_i and p_j are periodic then $\sigma\mathfrak{K}_i = \sigma\mathfrak{K}_j$ for Wada basins $\mathring{G}_i$ and $\mathring{G}_j$ respectively, or that too, there exist integers forming arithmetic progression $\mathfrak{A}$ arbitrary long, such that

$$\mathring{G}_i=\psi_a(\mathring{G}_j)$$

for every $a\in\mathfrak{A}$ □

5. B-saddle Colouring Example

Let us call the 3-separatrix fixed or periodic point *B-saddle*, by the *bike saddle* image.

Birkhoff curve containing the only B-saddle turns out to be four regions common boundary. One can prime example with rotary symmetry, due to it three Wada basins have identical the rotation number, in contrast to Wada ocean.

In paper [16], the authors did not solve the regions colouring problem, because ones solved other problems. Inter alia the possibility of constructing the Y-connection volt-ampere characteristic as the dynamic system action result has been studied with necessary properties.

Omitting the preliminary justifications and constructions (about this, see [16]) let us define the dynamical system by the diffeomorphism action on the plane at iterations as follows:

1° at first, one introduce the notations

$$\tilde{x} \overset{def}{=} \tanh(\varrho \cos \phi) \quad \text{and} \quad \tilde{y} \overset{def}{=} \tanh(\varrho \sin \phi);$$

2° Dynamic system defined by action of map in polar coordinates

$$\mathcal{Z}(\varrho, \phi) \mapsto \varrho \cdot \mathrm{e}^{i\phi}, \tag{9}$$

where $R = \pm 1$, $T = \pm 1$ and

$$\mathcal{Z}(\varrho, \phi) \overset{def}{=} T \left(\frac{R\tilde{x} + \tilde{y}}{p} + iq\tilde{y} \right) \mathrm{e}^{iR\varphi(\tilde{x}, \tilde{y})};$$

3° then for the mapping from the formula (9), the formula (by the simplest means) is constructed when moving in a positive direction

$$(|\mathcal{Z}(\varrho, \phi)|, \arg \mathcal{Z} + \Phi(\phi) + \Delta) \mapsto (\varrho, \phi) \tag{10}$$

and when moving in a negative direction

$$(|\mathcal{Z}(\varrho, \phi)|, \overline{\arg \mathcal{Z}} + \Phi(\phi) - \Delta) \mapsto (\varrho, \phi); \tag{11}$$

where $\Phi(\phi) \overset{def}{=} \phi - \phi \,(\mathrm{mod}\ 2\pi)$;

4° the formula for component $|\mathcal{Z}(\varrho, \phi)|$ is constructed as follows

$$|\mathcal{Z}(\varrho, \phi)| \overset{def}{=} \sqrt{\left(\frac{R\tilde{x} + \tilde{y}}{p} \right)^2 + q^2 y^2} \mapsto \varrho, \tag{12}$$

while components $\arg \mathcal{Z}$, if $R = 1$, and $\overline{\arg \mathcal{Z}}$, if $R = -1$, are defined by the formulae

$$\arg \mathcal{Z} \overset{def}{=} R\varphi(\tilde{x}, \tilde{y}) + \arctan \frac{qp\tilde{y}}{R\tilde{x} + \tilde{y}}, \tag{13}$$

$$\overline{\arg \mathcal{Z}} \overset{def}{=} R\varphi(\tilde{x}, \tilde{y}) - \arctan \frac{qp\tilde{y}}{R\tilde{x} + \tilde{y}}; \tag{14}$$

5° let us make a natural replacement of variables in the formulae (10) and (11)

$$\varrho \cdot \exp i\,(\phi/\Pi - \phi_0) \mapsto u + iv,$$

where $2 \cdot \Pi \in \mathbb{Z} \setminus \{0\}$, $\phi_0 \in \mathbb{R}$;

Then the maps acting at iterations are defined by the formulae when moving in the positive direction

$$\left(|\mathcal{Z}(\varrho, \phi)|, \frac{\arg \mathcal{Z} + \Phi(\phi) + \Delta}{\Pi} \right) \mapsto u + iv, \tag{15}$$

and when moving in the negative direction

$$\left(|\mathcal{Z}(\varrho, \phi)|, \frac{\overline{\arg \mathcal{Z}} + \Phi(\phi) - \Delta}{\Pi} \right) \mapsto u + iv; \tag{16}$$

6° therefore the replacement $(u, v) \mapsto (\varrho, \phi)$ occurs according to the formulae

$$\sqrt{u^2 + v^2} \mapsto \varrho, \quad \Pi \cdot (\phi_0 + \arctan(v/u) + m\pi) \mapsto \phi,$$

for all $m \in \mathbb{Z}$;

7° then the formulae (15) and (16) are rewritten in the following form

$$\left(|\mathcal{Z}(\varrho, \phi)|, \frac{\arg \mathcal{Z} + \Gamma(u, v) + \Delta}{\Pi} \right) \mapsto u + iv, \tag{17}$$

$$\left(|\mathcal{Z}(\varrho, \phi)|, \frac{\overline{\arg \mathcal{Z}} + \Gamma(u.v) - \Delta}{\Pi} \right) \mapsto u + iv, \tag{18}$$

where

$$\Gamma(u, v) \overset{def}{=} \Phi(\Pi \cdot (\phi_0 + \arctan(v/u))) + 2\pi\Pi \lfloor m/2 \rfloor;$$

8° the variables $\tilde{x}$ and $\tilde{y}$ are written as follows

$$\tilde{x} = \tan(\sqrt{u^2 + v^2} \cos(\Pi \cdot \omega(u, v))),$$
$$\tilde{y} = \tan(\sqrt{u^2 + v^2} \sin(\Pi \cdot \omega(u, v))),$$

where $\omega(u, v) \overset{def}{=} \phi_0 + \arctan(v/u) + m\pi$.

Now suppose $R = 1$ and $\Pi = 3/2$.

Remark 5 (on the technical details for the colouring). *Suppose for every invariant region its colour has been defined in PostScript codes, for own instance, in case of three Wada basins and Wada ocean ($\nu = 3$) as follows*

$$\begin{array}{c|cccl} \mathcal{K}_0 & r_0 & g_0 & b_0 & \text{setrgbcolor} \\ \mathcal{K}_1 & r_1 & g_1 & b_1 & \text{setrgbcolor} \\ \mathcal{K}_2 & r_2 & g_2 & b_2 & \text{setrgbcolor} \\ \mathcal{K}_3 & r_3 & g_3 & b_3 & \text{setrgbcolor} \\ \hline \overline{\mathcal{K}} & 1 & 1 & 1 & \text{(theorems condition)} \end{array} \tag{19}$$

or that too the boundary is summary coloured to be white colour.

Now one can colour the regions. There exist four colouring variants accurate to the angle of rotation.

Now back to the term ¡¡sufficiently large K¿¿. Suppose $\sigma\mathfrak{K}_1 = \ldots = \sigma\mathfrak{K}_\nu > \sigma\mathfrak{K}_0$ then

Theorem 6 *Suppose the rotations numbers such, that $\sigma\mathfrak{K}_l > \sigma\mathfrak{K}_0$ for colouring invariant regions G_0 and G_l respectively. Then there subsists the sequence $\mathfrak{F}_{0l}$ for every $l \in \overline{1,\nu}$, such that $\sigma\mathfrak{F}_{0l} = \sigma\mathfrak{K}_0 \colon \sigma\mathfrak{K}_l$.*

PROOF. Suppose $\mathcal{K}_0^{(0)} \subset G_0$ and $\mathcal{K}_l^{(0)} \subset G_l$, for all $l \in \overline{1,\nu}$ (defined by (7)), are colouring points sets from two different regions, such that there exists $\varepsilon_0 > 0$ for coverage $\Omega_{\varepsilon_0}(\Upsilon)$ and

$$\#(U(\psi_{k_\mu}(\xi_l)) \cap \mathcal{K}_0^{(0)}) \colon \#(U(\psi_{k_\mu}(\xi_l)) \cap \mathcal{K}_l^{(0)}) = 1.$$

Now let us will increase consistently the colouring points number in G_0 as follows

$$\#\mathcal{K}_0^{(1)} = 2\#\mathcal{K}_0^{(0)}, \ \ldots, \ \#\mathcal{K}_0^{(K)} = (K+1)\#\mathcal{K}_0^{(0)}, \ \ldots$$

being an increasing arithmetic progression. Therefore there exists decreasing sequence

$$\varepsilon_1, \ \varepsilon_2, \ \ldots, \ \varepsilon_K, \ \ldots,$$

such that the following equalities

$$\frac{\#(U(\psi_{k_\mu}(\xi_l)) \cap \mathcal{K}_0^{(K)})}{\#(U(\psi_{k_\mu}(\xi_l)) \cap \mathcal{K}_l^{(K)})} = 1. \tag{20}$$

are faithful for all $K \in \mathbb{N}$. Then increasing sequence

$$\#\mathcal{K}_l^{(0)}, \ \#\mathcal{K}_l^{(1)}, \ \ldots, \ \#\mathcal{K}_l^{(K)}, \ \ldots$$

in combination with own majoritarian increasing sequence

$$\#\mathcal{K}_l^{(0)} \cdot \left(1, \ \left\lceil \frac{\#\mathcal{K}_l^{(1)}}{\#\mathcal{K}_l^{(0)}} \right\rceil, \ \ldots, \ \left\lceil \frac{\#\mathcal{K}_l^{(K)}}{\#\mathcal{K}_l^{(0)}} \right\rceil, \ \ldots \right).$$

deliver the sequence

$$\mathfrak{F}_{0l}\colon \quad 1, \ \left\lceil \frac{\#\mathcal{K}_l^{(1)}}{\#\mathcal{K}_l^{(0)}} \right\rceil, \ \ldots, \ \left\lceil \frac{\#\mathcal{K}_l^{(K)}}{\#\mathcal{K}_l^{(0)}} \right\rceil, \ \ldots \tag{21}$$

Thus from condition (20) in combination with the theorem 5 the result is $\sigma\mathfrak{F}_{0l} = \sigma\mathfrak{K}_0 \colon \sigma\mathfrak{K}_l$ □

Corollary 2 *If $\sigma\mathfrak{K}_l = \sigma\mathfrak{K}_0$ then $\mathfrak{F}_{0l} \equiv \mathbb{N}$.*

PROOF. Indeed, if $\sigma\mathfrak{K}_l = \sigma\mathfrak{K}_0$ then

$$\#\mathcal{K}_0^{(K)} = \#\mathcal{K}_l^{(K)}$$

for all $K \in \mathbb{N}$. Therefore $\mathfrak{F}_{0l} \equiv \mathbb{N}$ and $\sigma\mathfrak{F}_{0l} \equiv 1$ □

Corollary 3 *For any Birkhoff curve Υ, there subsists at least one sequence $\mathfrak{F}_{0l}$ such, that*

$$0 < \sigma\mathfrak{F}_{0l} < 1.$$

However into practice far away not always one can to definitely assert either $\sigma\mathfrak{K}_l > \sigma\mathfrak{K}_0$ or $\sigma\mathfrak{K}_l < \sigma\mathfrak{K}_0$, and even one can not to define either $\sigma\mathfrak{K}_l \neq \sigma\mathfrak{K}_0$ or $\sigma\mathfrak{K}_l = \sigma\mathfrak{K}_0$. Nevertheless in such an uncertain situation, for any pair of regions G_i and G_j the desire to find a sequence of type $\mathfrak{F}_{ij}$ remains relevant regardless of whether $\sigma\mathfrak{K}_l \neq \sigma\mathfrak{K}_0$ or $\sigma\mathfrak{K}_l = \sigma\mathfrak{K}_0$.

Theorem 7 *For any pair of invariant regions G_i and G_j there exists increasing sequence $\mathfrak{F}_{ij}$, such that*

$$\sigma\mathfrak{F}_{ij} = \sigma\mathfrak{K}_i \colon \sigma\mathfrak{K}_j$$

for all $i, j \in \overline{0,\nu}$, if the following equality

$$\frac{\#(U(\psi_{k_\mu}(\xi_l)) \cap \mathcal{K}_i^{(K)})}{\#(U(\psi_{k_\mu}(\xi_l)) \cap \mathcal{K}_j^{(K)})} = 1. \tag{22}$$

is faithful. Moreover increasing sequence $\mathfrak{F}_{ij}$ does not depend from the construction method.

PROOF. The cases of $\sigma\mathfrak{K}_i > \sigma\mathfrak{K}_j$ and $\sigma\mathfrak{K}_i = \sigma\mathfrak{K}_j$ have been considered in the proofs of the theorem 6 and corollary 2. Now suppose

$$\#\mathcal{K}_j^{(1)} = 2\#\mathcal{K}_j^{(0)}, \ \ldots, \ \#\mathcal{K}_j^{(K)} = (K+1)\#\mathcal{K}_j^{(0)}, \ \ldots$$

be an increasing arithmetic progression. Then there exists decreasing sequence

$$\varepsilon_1, \ \varepsilon_2, \ \ldots, \ \varepsilon_K, \ \ldots,$$

such that equality (22) is faithful for all $K \in \mathbb{N}$. Then every element of the arithmetic progression corresponds to an element of the increasing sequence

$$\#\mathcal{K}_i^{(0)}, \ \#\mathcal{K}_i^{(1)}, \ \ldots, \ \#\mathcal{K}_i^{(K)}, \ \ldots$$

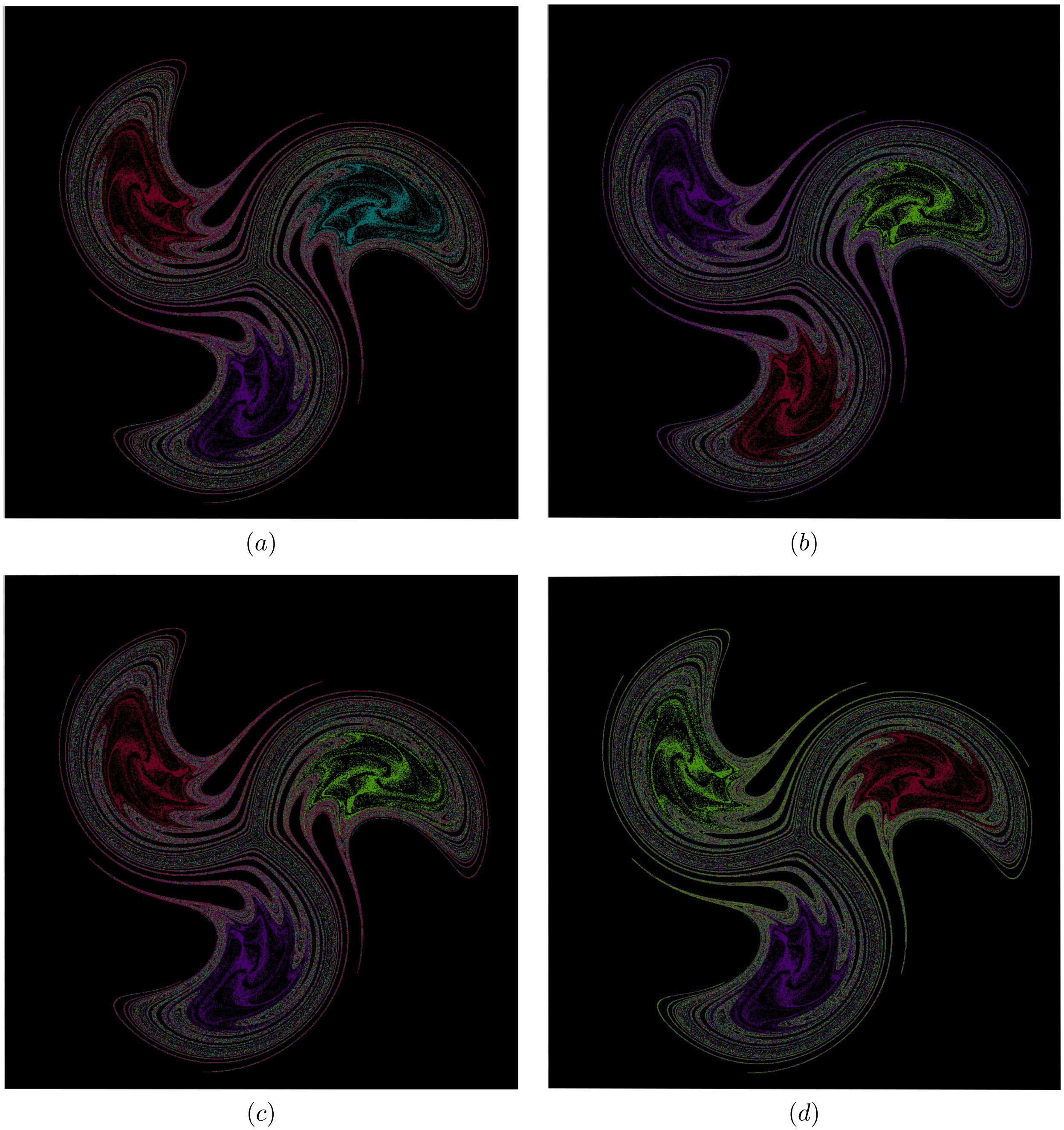

Figure 1: Invariant colourings for three Wada basins and an Wada ocean with a common boundary being Birkhoff curve having the only fixed point being inverse B - saddle for the dissipative action ψ at $\Pi = 3/2$ and $R = 1$, defined by formula (17), in relation $2:4:4:5$.

6. Conclusion

The source of the ergodic theory for Wada basins served from a remark in recent article [4]. Proceed on empirical and intuitive considerations, the author made estimates of the relations of the colouring densities of invariant regions, in order to their common border turns to be discoloured (i.e. white). The problem solution of the ¡¡boundary discolouration¿¿ is turned out to be directly related to the Wada basins ergodic properties.

The circle diffeomorphisms with irrational rotation number torn out to be in a certain sense simple rotations, or more exactly.

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