The dynamical systems (A, G, ω) and pseudo-differential operators on locally compact commutative groups

MYKOLA YAREMENKO
Department of mathematics
The National Technical University of Ukraine,
"Igor Sikorsky Kyiv Polytechnic Institute", Kyiv, Ukraine
UKRAINE

Abstract: - In the first part, we consider dynamical systems (A, G, ω) locally compact Abelian group, we generalize the Takesaki-Takai theorem and show that the τ-double-crossed product $\left(\mathrm{A} \times_{\omega}^{\tau} G\right) \times_{\hat{\omega}}^{\tau} \hat{G}$ is isomorphic to the tensor product $\mathrm{A} \otimes LK\left(L^2(G)\right)$, where the space $LK\left(L^2(G)\right)$ of all compact operators on $L^2(G)$. In the second part, we introduce the $\mathfrak{I}$- Wigner function, which generates pseudo-differential operators $Op_{\mathfrak{I}}(a)$ by association $\left\langle Op_{\mathfrak{I}}(a)\psi, \varphi\right\rangle_{L^2(G)} = \left\langle \hat{a}, W_{\mathfrak{I}}(\psi, \varphi)\right\rangle_{B^2(\hat{G}\times G)}$. We establish that assume $Op_{\mathfrak{I}}(a)$ is a bounded pseudo-differential operator $L^2(G) \otimes B^2(\hat{G}) \to B\left(L^2(G)\right)$ then operator $Op_{\mathfrak{I}}(a)$ extends to topological isomorphism $\wp'(G) \otimes \left(F\left(\wp(\hat{G})\right)\right)' \to B\left(\wp(G), \wp'(G)\right)$, where $\wp(G)$ is the Bruhat space $\wp(G)$.

Key-Words: - Takesaki-Takai duality, window Fourier transform, Wigner function, ambiguity function, C^*-algebra, Pontryagin duality, pseudo-differential operator.

Received: December 8, 2022. Revised: July 19, 2023. Accepted: August 25, 2023. Published: September 27, 2023.

1 Introduction

Quantum modeling represents physical observables by linear operators on separable Hilbert space. The Stone-Neumann theorem shows the unitary equivalency of any pair of irreducible representations of the canonical commutation of the linear operators that satisfy the Weyl condition, so, one-parameter unitary groups correspond to self-adjoint operators on a Hilbert space. In the quantum framework, essential observables are position and momentum that are governed by the Schrodinger quantum description representation of a particle if we consider a single particle model. The second Stone-Neumann shows that in the Schrodinger left-translation representation model, the C^*-algebra of compact operators on the separable Hilbert space is isomorphic to $C_{\mathbf{0}}(G) \times_{Lt} G$.

In the present article, we generalized the Takai duality theory to include crossed-product C^*-algebras $\mathrm{A} \times_{\omega}^{\tau} G$ as the completion of $\Psi \in C_C(G, \mathrm{A})$ with respect to the enveloping norm $\|\Psi\|_{env} = \sup_{(u,\rho)} \left\|\left(\rho \times_{\omega}^{\tau} u\right)(\Psi)\right\|$. Thus, we establish the existence of the isomorphism between the double-crossed product $\left(\mathrm{A} \times_{\omega}^{\tau} G\right) \times_{\hat{\omega}}^{\tau} \hat{G}$ and the tensor product $\mathrm{A} \otimes LK\left(L^2(G)\right)$. The Takai duality can be seen as an extension of the Pontryagin duality, which establishes isomorphism between the locally compact Abelian-Hausdorff group G and its double dual $\hat{\hat{G}}$ and guarantees the Fourier inversion formula and Plancherel theorem that are necessary for proving the Takai theory.

In 2022, Zhen Wang and Sen Zhu solved the Phillips problem related to the L^p-variant of the Takai theory, in 2014. Zhen Wang and Sen Zhu showed that double-crossed product $Env_{\hat{\omega}}\left(L^p\left(\hat{G}, Env_{\omega}\left(L^p\left(G,\mathrm{A}\right)\right)\right)\right)$ is isomorphic to the tensor product $\mathrm{A}\otimes LK\left(L^p\left(G\right)\right)$ if and only if either $p=2$ or the group G is finite.

Employing Pontrygin duality and the Fourier transform on the locally compact commutative group G, we introduce the $\mathfrak{I}$-Wigner function and quantize some wide classes of pseudo-differential operators corresponding to symbols on the Pontrygin pairs. We establish a natural correspondence between classes of symbols and pseudo-differential operators on locally compact commutative groups. We consider new types of pseudo-differential operators $Op_{\mathfrak{I}}\left(a\right): L^2\left(G\right)\to L^2\left(G\right)$ and $Op_{\mathfrak{I}}\left(\tilde{a}\right): L^2\left(G\right)\to L^2\left(G\right)$ associated by

$$\left\langle Op_{\mathfrak{I}}\left(a\right)\psi,\varphi\right\rangle_{L^2(G)}=\left\langle \hat{a}, W_{\mathfrak{I}}\left(\psi,\varphi\right)\right\rangle_{B^2\left(\hat{G}\times G\right)},$$

$$\hat{a}\left(\chi,h\right)=\int_{\hat{G}}\int_{G}\overline{\chi\left(g\right)}\xi\left(h\right)a\left(g,\xi\right)d\mu\left(g\right)d\hat{\mu}\left(\xi\right),$$

if we consider $Op_{\mathfrak{I}}$ as an integral operator then ist kernel is given by

$$K_{\mathfrak{I}}\left(a\right)\left(g,h\right)=$$

$$\int_{\hat{G}}Trac_{\chi}\left(\chi\left(h^{-1}g\right)a\left(\tau\left(g,h\right),\chi\right)\right)d\hat{\mu}\left(\chi\right)$$

and

$$\left\langle Op_{\mathfrak{I}}\left(\tilde{a}\right)\psi,\varphi\right\rangle_{L^2(G)}=\left\langle \tilde{a}, W_{\mathfrak{I}}\left(\psi,\varphi\right)\right\rangle_{B^2\left(\hat{G}\times G\right)}$$

for all $\psi,\varphi\in L^2\left(G\right)$, where we introduce the $\mathfrak{I}$-Wigner function $W_{\mathfrak{I}}: L^2\left(G\right)\times\overline{L^2\left(G\right)}\to L^2\left(G\times\hat{G}\right)$ by

$$W_{\mathfrak{I}}\left(\psi,\varphi\right)\left(g,\xi\right)=$$

$$\int_{G}\overline{\xi\left(h\right)}\psi\left(\tau_1\left(g,h\right)\right)\overline{\varphi\left(\tau_2\left(g,h\right)\right)}d\mu\left(h\right).$$

We establish that a bounded pseudo-differential operator $Op_{\mathfrak{I}}\left(a\right)$ mapping $L^2\left(G\right)\otimes B^2\left(\hat{G}\right)\to B\left(L^2\left(G\right)\right)$ can be extended to isomorphism

$$\wp'\left(G\right)\otimes\left(F\left(\wp\left(\hat{G}\right)\right)\right)'\to B\left(\wp\left(G\right),\wp'\left(G\right)\right),$$

where $\left(F\left(\wp\left(\hat{G}\right)\right)\right)'$ is the space of all linear functionals on $F\left(\wp\left(\hat{G}\right)\right)$.

The quantization of pseudo-differential operators has been actively studied since the second half of the last century therefore the literature is too extensive to be properly viewed in the present work some modern perspectives can be found in [1 -25].

2. Pseudo-differential operators and dynamic systems

Let G be a locally compact Hausdorff-Abelian group equipped with the Radon-Haar measure μ. Let $\hat{G}$ be the Pontryagin dual of G with dual measure $\hat{\mu}$.

The Fourier transform F is defined by

$$F\left(\psi\right)\left(\chi\right)=\hat{\psi}\left(\chi\right)=\int_{G}\overline{\chi\left(g\right)}d\mu\left(g\right)$$

for all $\chi\in\hat{G}$ and $\psi\in L^1\left(G,\mu\right)$. The Fourier transform extends from $L^2\left(G,\mu\right)\cap L^1\left(G,\mu\right)$ to an unitary isomorphism from $L^2\left(G,\mu\right)$ to $L^2\left(\hat{G},\hat{\mu}\right)$.

The pseudo-differential operator $Op\left(a\right):\psi\to A\left(\psi\right)$ with a symbol $a\in L^{\infty}\left(G\times\hat{G}\right)$ is presented by

$$\left(Op\left(a\right)\psi\right)\left(g\right)=\int_{\hat{G}}\xi\left(g\right)a\left(g,\xi\right)\hat{\psi}\left(\xi\right)d\hat{\mu}\left(\xi\right)$$

for all $g\in G$.

We can rewrite the pseudo-differential operator on the locally compact group G in the following form

$$\left(Op\left(a\right)\psi\right)\left(g\right)=\int_{\hat{G}}\xi\left(g\right)a\left(g,\xi\right)\hat{\psi}\left(\xi\right)d\hat{\mu}\left(\xi\right)=$$

$$=\int_{\hat{G}}\int_{G}\xi\left(g\right)a\left(g,\xi\right)\overline{\xi\left(h\right)}\psi\left(h\right)d\mu\left(h\right)d\hat{\mu}\left(\xi\right)=$$

$$= \int_{\hat{G}}\int_{G}\int_{G}\int_{\hat{G}} \chi(g)\xi(g)\overline{\xi(k)}\hat{a}(\chi,k)\overline{\xi(h)}\psi(h)\times$$
$$d\hat{\mu}(\chi)d\mu(k)d\mu(h)d\hat{\mu}(\xi)=$$
$$= \int_{G\times\hat{G}} \hat{a}(\chi,h)m_{\chi}\tau_{h^{-1}}\psi(g)d\hat{\mu}(\chi)d\mu(h)$$

for all $g \in G$, where we denote

$$\hat{a}(\chi,h) = \int_{\hat{G}}\int_{G} \overline{\chi(g)}\xi(h)a(g,\xi)d\mu(g)d\hat{\mu}(\xi),$$

and $\tau_h\psi(g) = \psi(h^{-1}g)$ is the translation and $m_\chi\psi(g) = \chi(g)\psi(g)$ is modulation operators.

Let A be a C^*-algebra. Let (A, G, ω) be triplet such that mapping $\omega : G \to Aut(\mathsf{A})$ is a strongly continuous automorphism, namely, for each $g \in G$ mapping $\omega(g) : \mathsf{A} \to \mathsf{A}$ is a C^*-algebra automorphism, and for each $g \in G$, the mapping $g \mapsto \omega(g)(\psi) \in \mathsf{A}$ is continuous for all $\psi \in \mathsf{A}$ and such that $\omega(g) \circ \omega(h) = \omega(gh)$ for all $g, h \in G$. Such a triplet (A, G, ω) is called a dynamical system.

Let H be a Hilbert space and let $K(H)$ be the C^*-algebra of all compact operators $H \to H$, and $B(H)$ be a space of all bounded operators $H \to H$.

We denote by $u : G \to U(H)$ a strongly continuous unitary representation, where the set $U(H)$ is a space of all unitary operators $H \to H$ and $\rho : \mathsf{A} \to B(H)$ is a non-degenerate representation of the C^*-algebra A in H.

If the equality

$$u(g)\rho(\psi)u(g)^* = \rho(\omega(g)(\psi))$$

holds for all $g \in G$ and all $\psi \in \mathsf{A}$ then a triplet (ρ, u, H) is called a covariant representation of the dynamical system (A, G, ω).

Example. Let G be a locally compact Abelian group then the C^*-algebra $\mathsf{A} = \tilde{C}_0(G)$ consist of all bounded uniformly continuous functions defined on G, a strongly continuous automorphism $\omega : G \to Aut(\tilde{C}_0(G))$ is given by $\omega(h,\psi)(g) = \omega(h^{-1}g)$. We take a Hilbert space H as $L^2(G,\mu)$ and a strongly continuous unitary representation $u : G \to U(L^2(G,\mu))$ is given by $u(h,\phi)(g) = \phi(h^{-1}g)$, and a representation of the C^*-algebra $\tilde{C}_0(G)$ in $L^2(G,\mu)$ given by $\rho : \tilde{C}_0(G) \to B(L^2(G,\mu))$, $\rho(\psi)\phi = \psi\phi = m_\psi\phi$. So defined system is called the Schrodinger representation.

3. The crossed product of C^*-algebras

Let (A, G, ω) be a dynamical system and let $\tau : G \to G$ be a continuous function.

For all $\Psi, \Upsilon \in C_C(G, \mathsf{A})$, we define a binary operation $\Diamond_\tau$ by

$$(\Psi \Diamond_\tau \Upsilon)(g) = \int_G \omega\left(\tau(g)^{-1}\tau(h), \Psi(h)\right)\times$$
$$\omega\left(\tau(g)^{-1}h\tau(h^{-1}g), \Upsilon(h^{-1}g)\right)d\mu(h)$$

and an adjoint element $\Psi^{\Diamond_\tau}(g) : G \to \mathsf{A}$ by

$$\Psi^{\Diamond_\tau}(g) = \omega\left(\tau(g)^{-1}g\tau(g^{-1}), \left(\Psi(g^{-1})\right)^*\right)$$

for all $g \in G$.

If we define the norm

$$\|\Psi\| = \int_G \|\Psi(g)\|_{\mathsf{A}}\, d\mu(g)$$

then vector space $C_C(G, \mathsf{A})$ becomes a Banach algebra with operations $\Diamond_\tau$ and adjoint $\Psi^{\Diamond_\tau}$.

Definition 1. ***We define a homomorphism*** $\rho \times_\omega^\tau u : C_C(G, \mathsf{A}) \to B(H)$ ***by***

$$\left(\rho \times_\omega^\tau u\right)(\Psi) = \int_G \rho\left(\omega(\tau(g), \Psi(g))\right)u(g)d\mu(g)$$

for all $\Psi \in C_C(G, \mathsf{A})$.

Definition 2. ***The enveloping norm of the*** $\Psi \in C_C(G, \mathsf{A})$ ***is given by***

$$\|\Psi\|_{env} = \sup_{(u,\rho)} \left\|\left(\rho \times_\omega^\tau u\right)(\Psi)\right\|.$$

The crossed-product C^****-algebra*** $\mathsf{A} \times_\omega^\tau G$ ***is the completion*** $C_C(G, \mathsf{A})$ ***with respect to the enveloping norm.***

For all $\Psi \in C_C(G, \mathrm{A})$ straightforward consideration yields

$$\|\Psi\|_{env} \le \|\Psi\| = \int_G \|\Psi(g)\|_{\mathrm{A}} \, d\mu(g).$$

Definition 3. ***The dual C^*-action $\hat{\omega}: C_C(G, \mathrm{A}) \to \mathrm{A} \times_\omega G$ is defined by the formula***

$$\hat{\omega}(\xi, \Psi)(g) = \overline{\xi(g)} \Psi(g)$$

for all $g \in G$ and $\xi \in \hat{G}$.

Example. In the case of the Schrodinger representation, we have

$$\rho \times_\omega^\tau u : L^1\left(G, \tilde{C}_0(G)\right) \to B\left(L^2(G, \mu)\right).$$

Statement 1. ***Let (A, G, ω) be a dynamical system then the mapping $\hat{\omega}: C_C(G, \mathrm{A}) \to \mathrm{A} \times_\omega G$ is an authorphism of $\mathrm{A} \times_\omega G$ for each character $\xi \in \hat{G}$.***

Proof. Assume $\Psi, \Upsilon \in C_C(G, \mathrm{A})$ then we calculate

$$\hat{\omega}(\xi, \Psi \Diamond \Upsilon)(g) = \overline{\xi(g)} (\Psi \Diamond \Upsilon)(g) =$$
$$= \int_G \overline{\xi(g)} \Psi(h) \omega\left(h, \Upsilon\left(h^{-1} g\right)\right) d\mu(h) =$$
$$= \int_G \hat{\omega}(\xi, \Psi(h)) \omega\left(h, \hat{\omega}\left(\xi, \Upsilon\left(h^{-1} g\right)\right)\right) d\mu(h) =$$
$$= \left(\hat{\omega}(\xi, \Psi) \Diamond \hat{\omega}(\xi, \Upsilon)\right)(g)$$

for all $g \in G$ and $\xi \in \hat{G}$.

Next, we consider adjoint elements, so that

$$\hat{\omega}\left(\xi, \Psi^\Diamond\right)(g) = \overline{\xi(g)} \Psi^\Diamond(g) =$$
$$= \overline{\xi(g)} \omega\left(g, \left(\Psi\left(g^{-1}\right)\right)^\Diamond\right) =$$
$$= \omega\left(g, \left(\overline{\xi\left(g^{-1}\right)} \Psi\left(g^{-1}\right)\right)^\Diamond\right) =$$
$$= \omega\left(g, \hat{\omega}\left(\xi, \Psi\left(g^{-1}\right)^\Diamond\right)\right) =$$
$$= \hat{\omega}\left(\xi, \Psi^\Diamond(g)\right)$$

for all $g \in G$.

Statement 2. ***Let (A, G, ω) be a dynamical system then the mapping $\hat{\omega}: \hat{G} \to \mathrm{A} \times_\omega G$ is a group homomorphism.***

Proof. We show that mapping $\hat{\omega}$ is a group homomorphism, assuming $\chi, \xi \in \hat{G}$ we obtain the equality

$$\hat{\omega}(\chi\xi, \Psi)(g) = \left(\overline{\chi\xi}\right)(g) \Psi(g) =$$
$$= \overline{\chi}(g) \hat{\omega}(\xi, \Psi)(g) =$$
$$= \hat{\omega}(\chi, \hat{\omega}(\xi, \Psi))(g) = \left(\hat{\omega}(\chi) \hat{\omega}(\xi)\right)(\Psi)(g)$$

holds for all $\Psi \in C_C(G, \mathrm{A})$ and all $g \in G$.

For each $\xi \in \hat{G}$, there exists a neighborhood $E \subset \hat{G}$ of ξ such that all $\chi \in E$ we have

$$\|\hat{\omega}(\xi, \Psi) - \hat{\omega}(\chi, \Psi)\| =$$
$$\int_G \left|\overline{\xi}(g) - \overline{\chi}(g)\right| \|\Psi(g)\| d\mu(g) \le$$
$$\le \sup_{g \in \sigma(\Psi)} \left|\overline{\xi}(g) - \overline{\chi}(g)\right| \le \varepsilon$$

so that $\|\hat{\omega}(\xi, \Psi) - \hat{\omega}(\chi, \Psi)\|_{env} \le \varepsilon$ where $\sigma(\Psi)$ is the support of Ψ.

4. The Takesaki-Takai theory

Let G be a locally compact Hausdorff-Abelian group and (A, G, ω) be a dynamical system. So, we formulate the generalized variant of the Takesaki-Takai theorem.

Theorem (generalized Takesaki-Takai**) 1.** ***Let $LK\left(L^2(G)\right)$ be the space of all compact operators on $L^2(G)$ then there exists an isomorphism between $\left(\mathrm{A} \times_\omega^\tau G\right) \times_{\hat{\omega}}^\tau \hat{G}$ and maximal tensor product $\mathrm{A} \otimes LK\left(L^2(G)\right)$.***

Proof. Let (A, G, ω) be a dynamical system and let triplet $\left(\rho, u, L^2(G)\right)$ be a covariant representation of the dynamical system (A, G, ω). Then, by taking the completion of $C_C(G, \mathrm{A})$ with respect to the enveloping norm given by

$$\|\Psi\|_{env} = \sup_{(u, \rho)} \left\|\left(\rho \times_\omega^\tau u\right)(\Psi)\right\|,$$

we obtain the set $\left\{\mathrm{A} \times_\omega^\tau G, \quad \tau: G \to G\right\}$ of crossed-product C^*-algebras. For each pair (τ_1, τ_2), we define an isometric isomorphism $\Theta: \mathrm{A} \times_\omega^{\tau_1} G \to \mathrm{A} \times_\omega^{\tau_2} G$ given by

$$\Theta((\tau_1,\tau_2)(\Psi))(g)=\omega\left(\tau_2(g)^{-1}\tau_1(g),\Psi(g)\right)$$

for all $\Psi\in L^1(G,\mathrm{A})$. Straightforward calculations yield $\rho\times_{\omega}^{\tau_1}u=\left(\rho\times_{\omega}^{\tau_2}u\right)\circ\Theta(\tau_1,\tau_2)$ and an inverse element is given as $\Theta(\tau_1,\tau_2)^{-1}=\Theta(\tau_2,\tau_1)$.

By taking $\tau(g)=e$, we obtain the classical variant of Takai theory therefore the generalized Takesaki-Takai theorem is proven.

To complete our investigation we consider a classical variant of the Takai theorem.

For each $g\in G$, we define an automorphism $w(g,u)(\theta)=u(g)\theta u(g)^{*}$ for all $\theta\in LK\left(L^2(G)\right)$.

Theorem (Takai) 2. ***Let*** (A,G,ω) ***be a dynamical system, then there exists an isomorphism*** Υ ***between*** $(\mathrm{A}\times_{\omega}G)\times_{\hat{\omega}}\hat{G}$ ***and maximal tensor product*** $\mathrm{A}\otimes LK\left(L^2(G)\right)$***, this isomorphism is equivariant for*** $w\left(g,u^R\right)\otimes\omega$ ***and*** $\hat{\hat{\omega}}$***.***

The proof follows from the system of isomorphisms:

$$(\mathrm{A}\times_{\omega}G)\times_{\hat{\omega}}\hat{G}\xrightarrow{\Upsilon_1}\left(\mathrm{A}\times_{id}\hat{G}\right)\times_{id^{-1}\otimes\omega}G,$$

$$\left(\mathrm{A}\times_{id}\hat{G}\right)\times_{id^{-1}\otimes\omega}G\xrightarrow{\Upsilon_2}C_0(G,\mathrm{A})\times_{Lt\otimes\omega}G,$$

$$C_0(G,\mathrm{A})\times_{Lt\otimes\omega}G\xrightarrow{\Upsilon_3}C_0(G,\mathrm{A})\times_{Lt\otimes id}G,$$

$$C_0(G,\mathrm{A})\times_{Lt\otimes id}G\xrightarrow{\Upsilon_4}LK\left(L^2(G)\right)\otimes\mathrm{A},$$

therefore, the isomorphism of the Takai theorem can be found as a composition $\Upsilon=\Upsilon_4\circ\Upsilon_3\circ\Upsilon_2\circ\Upsilon_1$, where Lt is left translation. The final isomorphism Υ has a property $\Upsilon\left(\hat{\hat{\omega}}\right)=w\left(g,u^R\right)\otimes\omega$.

To accentuate the non-triviality of the generalized Takesaki-Takai theorem we formulate the Wang-Zhu results concerning the Takai duality for L^p-spaces. Let G be a countable discrete commutative group and A be a unital Hausdorff algebra of L^p-operators then $(\mathrm{A}\times_{\omega}G)\times_{\hat{\omega}}\hat{G}$ isomorphic $\mathrm{A}\otimes LK\left(L^p(G)\right)$ if and if either $p=2$ or group G is finite. The main problem arises in the analog of the mapping Υ_2.

5. The Rihaczek distribution

For all $h\in G$ and $\chi\in\hat{G}$, we denote translation and modulation operators by $\tau_h\psi(g)=\psi\left(h^{-1}g\right)$ and $m_{\chi}\psi(g)=\chi(g)\psi(g)$, respectively. So, equality $\tau_h m_{\chi}=\overline{\chi(h)}m_{\chi}\tau_h$ holds for all $h\in G$, $\chi\in\hat{G}$.

For all $\psi,\phi\in L^2(G,\mu)$, the Rihaczek distribution $R(\psi,\varphi)$ is given by

$$R(\psi,\varphi)(g,\chi)=\overline{\chi(g)}\psi(g)\overline{\hat{\varphi}(\chi)}$$

for all $(g,\chi)\in G\times\hat{G}$.

Statement 3. *Let* $\psi,\varphi\in M_v^1\left(G\times\hat{G}\right)$ *then*

$$\left\langle A_a m_{\chi}\tau_h\psi, m_{\xi}\tau_g\varphi\right\rangle_{L^2(G)}=$$

$$\overline{\chi\left(g^{-1}h\right)}\left(V_{R(\varphi,\psi)}a\right)\left((h,\xi),\left(\xi^{-1}\chi,g^{-1}h\right)\right),$$

where the window Fourier transform V_{ϕ} *is given*

$$V_{\varphi}\psi(g,\chi)=\int_G\psi(h)\overline{\varphi\left(g^{-1}h\right)}\overline{\chi(h)}\,d\mu(h).$$

Proof. For each pair $\psi,\varphi\in M_v^1\left(G\times\hat{G}\right)$, a pseudo-differential operator A_a corresponding with the symbol a can be rewritten in the form

$$\left\langle A_a(\psi)(\cdot),\varphi(\cdot)\right\rangle_{L^2(G)}=$$

$$=\int_G\int_{\hat{G}}\chi(h)a(h,\chi)\hat{\psi}(\chi)\overline{\varphi(h)}d\hat{\mu}(\chi)d\mu(h)=$$

$$=\left\langle a(\cdot,\cdot),R(\varphi,\psi)(\cdot,\cdot)\right\rangle_{L^2\left(G\times\hat{G}\right)},$$

so, we have

$$\left\langle A_a\left(m_{\chi}\tau_h\psi\right),m_{\xi}\tau_g\varphi\right\rangle_{L^2(G)}=$$

$$=\left\langle a(\cdot,\cdot),R\left(m_{\chi}\tau_h\psi,m_{\xi}\tau_g\varphi\right)(\cdot,\cdot)\right\rangle_{L^2\left(G\times\hat{G}\right)}=$$

$$=\int_G\int_{\hat{G}}\eta(k)a(k,\eta)\overline{m_{\chi}\tau_h\psi(k)}m_{\xi}\tau_g\varphi(\eta)\times$$

$$d\hat{\mu}(\eta)d\mu(k)=$$

$$=\overline{\chi\left(g^{-1}h\right)}\left(V_{R(\varphi,\psi)}a\right)\left((h,\xi),\left(\xi^{-1}\chi,g^{-1}h\right)\right).$$

The fundamental properties of pseudo-differential operators can be elucidated in terms of Modulation spaces $M^{\infty}\left(G\times\hat{G}\right)$.

Theorem 3. ***Let symbol*** $a \in M^{\infty}\left(G\times\hat{G}\right)$, ***let*** Λ ***be a quasi-lattice in*** $G\times\hat{G}$, ***and let*** $\left\{m_{\zeta}\tau_k\phi\right\}_{(k,\zeta)\in\Lambda}$ ***be a tight Gabor frame in*** $L^2(G)$ ***for*** $\phi\in M_v^1(G)$ ***with an admissible weight*** v. ***Then, there exists a function*** $\Theta\in L_v^1\left(G\times\hat{G}\right)$ ***such that***

$$\left|\left\langle A_a\left(m_{\chi}\tau_h\phi\right), m_{\xi}\tau_g\phi\right\rangle_{L^2(G)}\right|\le\Psi\left(h^{-1}g,\chi^{-1}\xi\right)$$

for all $(h,\chi),(g,\xi)\in G\times\hat{G}$ ***if and only if there exists a function*** $\theta\in\ell_v^1(\Lambda)$ ***such that***

$$\left|\left\langle A_a\left(m_{\zeta}\tau_k\phi\right), m_{\vartheta}\tau_s\phi\right\rangle_{L^2(G)}\right|\le\psi\left(k^{-1}s,\zeta^{-1}\vartheta\right)$$

for all $(k,\zeta),(s,\vartheta)\in\Lambda$.

The proof will follow from the expression

$$m_{\xi}\tau_g\phi=\sum_{(k,\zeta)\in\Lambda}\left\langle m_{\xi}\tau_g\phi, m_{\zeta}\tau_k\phi\right\rangle m_{\zeta}\tau_k\phi,$$

which follows from the tightness of the Gabor frame $\left\{m_{\zeta}\tau_k\phi\right\}_{(k,\zeta)\in\Lambda}$.

6. Pseudo-differential operators and $\Im$ -Wigner function

Let G be an amenable, locally compact Hausdorff group. The $\Im$-Wigner function $W_{\Im}:L^2(G)\times\overline{L^2(G)}\to L^2\left(G\times\hat{G}\right)$ is defined by

$$W_{\Im}(\psi,\varphi)(g,\xi)=\int_G\overline{\xi(h)}\psi\left(\tau_1(g,h)\right)\overline{\varphi\left(\tau_2(g,h)\right)}d\mu(h),$$

where functions $u=\tau_1(g,h),\quad v=\tau_2(g,h)$ such that

1) $\tau_1(g,e)=g,\quad\tau_2(g,e)=g$ for all $g\in G$;

2) for all $g\in G$, $h=uv^{-1}$ holds for all $(u,v)\in G\times G$;

3) the inverse mapping $\Im^{-1}:G\times G\to G\times G$ is given by

$$g=\tau(u,v)$$
$$h=uv^{-1},$$

for τ continuous mapping $G\times G\to G\times G$.

Let $\left\{H_{\xi},\quad\xi\in\hat{G}\right\}$ be a $\hat{\mu}$-measurable set of separable Hilbert spaces H_{ξ}. The direct integral is given by

$$B^2\left(\hat{G}\right)=\int_{\hat{G}}^{\oplus}H_{\xi}\otimes\overline{H}_{\xi}d\hat{\mu}(\xi)$$

and bi-linear integral form by

$$\left\langle\Psi,\Upsilon\right\rangle_{B^2(\hat{G})}=\int_{\hat{G}}^{\oplus}\left\langle\Psi(\xi),\Upsilon(\xi)\right\rangle_{B^2(H_{\xi})}d\hat{\mu}(\xi).$$

We define a pseudo-differential operator $Op_{\Im}(a):L^2(G)\to L^2(G)$ corresponding to the operator-valued symbol $a\in B^2\left(G\times\hat{G}\right)$ by associate this symbol with the integral form $\left\langle\hat{a},W_{\Im}(\psi,\varphi)\right\rangle_{B^2(\hat{G}\times G)}$ by

$$\left\langle Op_{\Im}(a)\psi,\varphi\right\rangle_{L^2(G)}=\left\langle\hat{a},W_{\Im}(\psi,\varphi)\right\rangle_{B^2(\hat{G}\times G)}$$

for all $\psi,\varphi\in L^2(G)$, where the Fourier transform is given by

$$\hat{a}(\chi,h)=\int_{\hat{G}}\int_G\overline{\chi(g)}\xi(h)a(g,\xi)d\mu(g)d\hat{\mu}(\xi).$$

Due to the Plancherel theorem, the pseudo-differential operator $Op_{\Im}(a)$ satisfies the estimation

$$\left|\left\langle Op_{\Im}(a)\psi,\varphi\right\rangle_{L^2(G)}\right|\le\|a\|_{B^2(\hat{G}\times G)}\|\psi\|_{L^2(G)}\|\varphi\|_{L^2(G)}$$

for all $\psi,\varphi\in L^2(G)$.

In our previous works, we showed that if an operator $A:L^2(G)\to L^2(G)$ is of the trace class, then, there exist sets $\{\psi_k\},\{\varphi_k\}\subset L^2(G)$ and $\{\lambda_k\}\subset C$, $\sum_k|\lambda_k|<\infty$ such that

$$A=\sum_k\lambda_k Op_{\Im}\left(W_{\Im}(\psi_k,\varphi_k)\right).$$

If we rewrite the pseudo-differential operator $Op_{\Im}(a)$ in the form

$$\left(Op_{\Im}(a)(\psi)\right)(g)=\int_G K_{\Im}(a)(g,h)\psi(h)d\mu(h),$$

then the kernel can be presented as

$$K_{\Im}(a)(g,h)=\int_{\hat{G}}Trac_{\chi}\left(\chi\left(h^{-1}g\right)a\left(\tau(g,h),\chi\right)\right)d\hat{\mu}(\chi).$$

Alternatively, we can define a pseudo-differential operator $Op_{\Im}(\tilde{a}) : L^2(G) \to L^2(G)$ by

$$\left\langle Op_{\Im}(\tilde{a})\psi, \varphi \right\rangle_{L^2(G)} = \left\langle \tilde{a}, W_{\Im}(\psi, \varphi) \right\rangle_{B^2(\hat{G}\times G)}$$

for all $\psi, \varphi \in L^2(G)$.

Theorem 4. ***Let the operator-valued symbol*** $\tilde{a}(\xi, g)$ ***satisfies the inequality***

$$\|\tilde{a}\|_{B^1(\hat{G}\times G)} = \int_{\hat{G}}\int_{G} \|\tilde{a}(\xi, g)\| d\mu(g) d\hat{\mu}(\xi) < \infty,$$

then the operator $Op_{\Im}(\tilde{a}) : L^2(G) \to L^2(G)$ ***satisfies the following estimation***

$$\|Op_{\Im}(\tilde{a})\| \le \|\tilde{a}\|_{B^1(\hat{G}\times G)},$$

so that the operator $Op_{\Im}(\tilde{a}) : L^2(G) \to L^2(G)$ ***is bounded.***

Proof. Applying the Plancherel theorem, we estimate

$$\left\langle Op_{\Im}(\tilde{a})\psi, \varphi \right\rangle_{L^2(G)} = \left\langle \tilde{a}, W_{\Im}(\psi, \varphi) \right\rangle_{B^2(\hat{G}\times G)} \le$$
$$\le \|\tilde{a}\|_{B^1(\hat{G}\times G)} \|\psi\|_{L^2(G)} \|\varphi\|_{L^2(G)},$$

which holds for all functions $\psi, \varphi \in L^2(G)$.

7. The kernel of pseudo-differential operators

Thus, we generalize the Weyl pseudo-differential calculus, so that $Op_{\Im} \leftrightarrow a$ by

$$(Op_{\Im}(a)(\psi))(g) =$$
$$\int_{G}\int_{\hat{G}} Trac_{\chi}\left(\chi(h^{-1}g) a(\tau(g,h), \chi)\right) d\mu(h) d\hat{\mu}(\chi),$$

$$K_{\Im}(a)(g,h) =$$
$$\int_{\hat{G}} Trac_{\chi}\left(\chi(h^{-1}g) a(\tau(g,h), \chi)\right) d\hat{\mu}(\chi)$$

for all $g, h \in G$, therefore, the identity operator corresponds to the unit symbol; and we expressed the symbol as

$$a(g,\xi) =$$
$$\int_{\hat{G}} Trac_{\chi}\left(\overline{\xi(h)} K_{\Im}(a)(\tau_1(g,h), \tau_2(g,h))\right) d\mu(h)$$

for all $g \in G, \quad \xi \in \hat{G}$.

We define the composition of pseudo-differential operators by

$$Op_{\Im}(a\#b) = Op_{\Im}(a) \circ Op_{\Im}(b)$$

so that operators' kernels are connected by

$$K_{\Im}(a\#b) = K_{\Im}(a) \#^{K} K_{\Im}(b).$$

The mapping $\psi \mapsto \left\langle Op_{\Im}(a)\psi, \varphi \right\rangle_{L^2(G)}$ defines a bounded linear functional on the linear space $L^2(G)$ so that there exist a mapping $Op_{\Im}(a)^*$ such that

$$\left\langle Op_{\Im}(a)\psi, \varphi \right\rangle_{L^2(G)} = \left\langle \psi, Op_{\Im}(a)^* \varphi \right\rangle_{L^2(G)}$$

for all $\psi, \varphi \in L^2(G)$, this mapping $Op_{\Im}(a)^*$ is called an adjoint operator.

Pseudo-differential operators corresponding to $B^2(G\times\hat{G})$-symbols are bounded operators from $L^2(G)$ to $L^2(G)$. If the group G is a differentiable manifold then the initial space can be chosen as the space of tempered distributions with compact supports. In the general case, we assume $\wp(G)$ is the Bruhat space and $\wp'(G)$ is a strong dual of $\wp(G)$. Standard arguments show that $\wp(G)$ is a dense subspace of $\wp'(G)$. A Bruhat space was introduced to extend the concept of space of C^∞ functions to include a wide class of locally compact Abelian groups.

Definition. ***The Bruhat space*** $\wp(G)$ ***consist of all functions*** $\psi \in L^\infty(G)$ ***such that there is a compact set*** $E(\psi) \subset G$ ***for which inequalities*** $\|\psi\|_{\infty,\ G\setminus E(\psi)^k} \le M(n) k^{-n}$ ***and*** $\|\hat{\psi}\|_{\infty,\ \hat{G}\setminus \hat{E}(\hat{\psi})^k} \le M(n) k^{-n}$ ***hold for all*** n***, each integer*** k ***and some constant*** $M(n)$***.***

The Bruhat space can be endowed with topology LF-space that generates by inequalities for $\wp(G)$, this topology coincides with the limit topology of $\wp(G)$.

Theorem 5. ***Let a pseudo-differential operator*** $Op_{\Im}(a)$ ***be bounded*** $L^2(G) \otimes B^2(\hat{G}) \to B(L^2(G))$ ***then*** $Op_{\Im}(a)$ ***extends to isomorphism***

$$\wp'(G) \otimes \left(F\left(\wp(\hat{G})\right)\right)' \to B\left(\wp(G), \wp'(G)\right),$$

where $\left(F\left(\wp\left(\hat{G}\right)\right)\right)'$ ***denotes the space of all linear functionals on*** $F\left(\wp\left(\hat{G}\right)\right)$ ***and*** F ***stands for the Fourier transform.***

The proof follows from the definitions.

4 Conclusion

In this article, we have established a new Takesaki-Takai theorem for the τ-double-crossed product $\left(\mathsf{A}\times_{\omega}^{\tau}G\right)\times_{\hat{\omega}}^{\tau}\hat{G}$. Also, we introduce the $\Im$- Wigner function and associate with it the class of pseudo-differential operators $Op_{\Im}(a)$, these operators can be extended from $L^2(G)\otimes B^2\left(\hat{G}\right)\to B\left(L^2(G)\right)$ to topological isomorphism on the Bruhat spaces $\wp(G)$. To establish additional properties of operators $Op_{\Im}(a)$ further studies are needed.

References:

[1] B. Abadie, Takai duality for crossed products by Hilbert C ∗ -bimodules, *J. Operator Theory* 64 (2010), 19–34.

[2] S. Albandik and R. Meyer, Product systems over Ore monodies, *Doc. Math.* 20, (2015) 1331–1402.

[3] A. Alldridge, C. Max, M. R. Zirnbauer, Bulk-Boundary Correspondence for Disordered Free-Fermion Topological Phases, *Commun. Math. Phys.* 377, 1761–1821, (2020).

[4] A. Benyi, K. Grochenig, C. Heil, and K. Okoudjou, Modulation spaces and a class of bounded multilinear pseudodifferential operators. *J. Operator Theory*, 54:389–401, (2005).

[5] E. Bedos, S. Kaliszewski, J. Quigg, and D. Robertson, A new look at crossed product correspondences and associated C ∗ –algebras, *J. Math. Anal. Appl.* 426, (2015), 1080– 1098.

[6] D. Blecher and N. C. Phillips, L p operator algebras with approximate identities I, *Pacific J. Math.* 303, (2019), no. 2, pp. 401-457.

[7] A. Carey, G. C. Thiang, The Fermi gerbe of Weyl semimetals, *Letters Math. Phys.* 111, 1-16, (2021).

[8] E. Cordero, K. Gr¨ochenig, F. Nicola and L. Rodino. Generalized Metaplectic Operators and the Schrodinger Equation with a Potential in the Sjostrand Class, *J. Math. Phys.*, 55(8):081506, 17, (2014)

[9] N. Dupuis, L. Canet, A. Eichhorn, W. Metzner, J. Pawlowski, M. Tissier, and N. Wschebor. The nonperturbative functional renormalization group and its applications, *Physics Reports* 910, 1–114, (2021).

[10] V. Deaconu, Group actions on graphs and C ∗ -correspondences, *Houston J. Math.* 44 (2018), 147–168.

[11] V. Deaconu, A. Kumjian, and J. Quigg, Group actions on topological graphs, *Ergodic Theory Dynam. Systems* 32, (2012),1527–1566.

[12] V. Fischer M. Ruzhansky, Quantization on nilpotent Lie groups, *Progress in Mathematics,* Birkhäuser Basel, (2016).

[13] H. Fujiwara and J. Ludwig, *Harmonic analysis on exponential solvable Lie groups*, Springer Monographs in Mathematics, Springer, Tokyo, (2015).

[14] E. Gardella and M. Lupini, Representations of étale groupoids on L p spaces, *Adv. Math.* 318, (2017), pp. 233-278.

[15] E. Gardella and H. Thiel, Representations of p-convolution algebras on L q -spaces. *Trans. Amer. Math.* Soc. 371, (2019), no. 3, pp. 2207-2236.

[16] S. Kaliszewski, J. Quigg and D. Robertson, Coactions on Cuntz-Pimsner algebras, *Math. Scand.* 116, (2015), 222–249.

[17] E. Katsoulis, *Non-selfadjoint operator algebras: dynamics, classification, and C ∗ -envelopes, Recent advances in operator theory and operator algebras,* 27–81, CRC Press, Boca Raton, FL, (2018).

[18] E. Katsoulis, C∗ -envelopes and the Hao-Ng Isomorphism for discrete groups, *International Mathematics Research Notices,* Volume 2017, Issue 18, (2017), 5751–5768.

[19] I. Raeburn. Dynamical systems and operator algebras. In National Symposium on Functional Analysis, *Optimization, and Applications,* pages 109–119. Australian National University, Mathematical Sciences Institute, (1999).

[20] M. Ruzhansky and V. Turunen, Global quantization of pseudo-differential operators on compact Lie groups, SU(2), 3-sphere, and homogeneous spaces, *Int. Math. Res. Not. IMRN* 11, (2013), 2439–2496.

[21] S. Sundar, C∗-algebras associated to topological Ore semigroups, Munster *J. of Math.* 9, (2016), no. 1, 155–185.

[22] M.I. Yaremenko Calderon-Zygmund Operators and Singular Integrals, *Applied Mathematics &*

Information Sciences: Vol. 15: Iss. 1, Article 13, (2021).

[23] Z. Wang and Y. Zeng, Gelfand theory of reduced group L p operator algebra, *Ann. Funct. Anal.* 13, 14, (2022).

[24] Z. Wang, S. Zhu, *On the Takai duality for L p operator crossed products,* preprint arXiv:2212.00408, (2022).

[25] M. I. Yaremenko, Sequences of the Projection-valued Measures and Functional Calculi. *Journal of Science and Mathematics Letters,* 11(2), 39–47, (2023).

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The author contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

No funding was received for conducting this study.

Conflict of Interest

The author has no conflict of interest to declare that is relevant to the content of this article.