# Trace class in separable reflexive Banach spaces, Lidskii theorem

MYKOLA YAREMENKO
Department of Partial Differential Equations,
The National Technical University of Ukraine,
"Igor Sikorsky Kyiv Polytechnic Institute", Kyiv, UKRAINE,

*Abstract:* - In this article, we generalize definitions of the trace class operators for the reflexive Banach spaces and develop the Lidskii theory for Hilbert spaces to incorporate a larger class of reflexive Banach spaces for which the analog of trace class operators can be introduced. The approximation for the compact operator $A \in B_C(X, Y)$ on reflexive Banach spaces by the set of finite rank operators $B_{Ran}(X, Y)$ is proposed and its properties are established.

*Key-Words:* - Reflexive Banach space, finite rank operator, trace class, trace operator, spectral theory, Gaussian distribution, Hahn-Banach theorem.

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## 1 Introduction (some definitions and notations)

Historically, the important motivation for the development of the theory of linear operators in Banach spaces was their application to the theory of the integral equations so Fredholm integral equations engender development of the theory of compact operators on the abstract reflexive Banach spaces, especially, when Banach space is self-adjoint functional Hilbert space and operators are of the Schrodinger type. Crucial characterization of such operators is a property of preservation of positivity of the bilinear form such operators often arise in quantum information science as quantum density operators due to the correspondence principle.

This article is dedicated to the trace class of linear operators in the reflexive Banach spaces. Although, there is extensive literature on the geometrical method of the reflexive Banach spaces and the theory of linear operator in such spaces [1-40], however, some questions that relate to spectral theory and trace operators need to be clarified.

Let $X$ and $Y$ be a pair of reflexive Banach spaces and let $A$ be a linear operator from $X$ to $Y$. The operator $A: X \to Y$ is called a finite rank operator or has a finite rank if the dimension of the image of the operator $A$ is a finite number.

Let $y \in Y$ and $\tilde{x}^* \in X^*$, by definition, the linear map $y \otimes \tilde{x}^* : X \to Y$ is

$$y \otimes \tilde{x}^*(x) = \tilde{x}^*(x)\, y,$$

so that

$$\left\| y \otimes \tilde{x}^* \right\| = \sup_{\|x\| \le 1} \left\| y \otimes \tilde{x}^* \right\| = \sup_{\|x\| \le 1} \left\| \tilde{x}^*(x)\, y \right\| =$$
$$= \|y\| \sup_{\|x\| \le 1} \left\| \tilde{x}^*(x) \right\| = \|y\| \left\| \tilde{x}^* \right\|$$

and $\left( y \otimes \tilde{x}^* \right) \subseteq span\{y\}$, so $y \otimes \tilde{x}^*$ is a finite rank operator.

**Definition 1.** ***For every element $x \in X$, we define an adjoint $x^* \in X^*$ as an element for which achieves the following equality***

$$\left\langle x, x^* \right\rangle = \|x\| \left\| x^* \right\|.$$

***Next, let us assume that $\{x_i\}$ and $\{x_i^*\}$ are two bases of reflexive Banach spaces $X$ and $X^*$, respectively, and let basis $\{x_i\}$ be orthonormal to $\{x_i^*\}$ in the following sense:***

$$\left\langle x_k, x_k^* \right\rangle = \|x_k\| \left\| x_k^* \right\| = 1$$

***for any $k$ and***

$$\left\langle x_i, x_j^* \right\rangle = 0$$

***for all $i \neq j$.***

**Definition 2.** ***A map $P: X \to X$ that satisfies the condition $PP \equiv P$ is called a projection (or the projection map or operator of projection).***

***Let linear operator $A$ maps from $X$ to $Y$, where $X$ , and $Y$ are a pair of reflexive Banach spaces. We denote $\{x_k\}$ and $\{y_k^*\}$ the bases of space $X$ and $Y^*$, respectively.***

***The trace $Trace(A)$ of the operator $A$ is***

$$Trace(A) = \sum_k \langle A(x_k), y_k^* \rangle,$$

***if this sum is independent of the choice of the bases.***

**Definition 3.** ***A pair of sets $X_1 \subset X$ and $X_1^* \subset X^*$ is called orthogonal if the equality***

$$\langle x, \tilde{x}^* \rangle = 0$$

***holds for arbitrary $x \in X_1$ and for arbitrary $\tilde{x}^* \in X_1^*$.***

**Definition 4.** ***Assume $X_1$ is a subspace of a Banach space $X$ and $\tilde{X}_1^*$ is a subspace of its adjoint $X^*$. Their annihilators $X_1^{\perp}$ and $\left(\tilde{X}_1^*\right)^{\perp}$ are defined by formulae***

$$X_1^{\perp} = \left\{ x^* \in X^* : \langle x, x^* \rangle = 0 \quad \forall x \in X_1 \right\}$$

***and, similarly***

$$\left(\tilde{X}_1^*\right)^{\perp} = \left\{ x \in X : \langle x, x^* \rangle = 0 \quad \forall x \in \tilde{X}_1^* \right\}.$$

**Definition 5.** ***The subspace $X_1^{\perp}$ is a set of all bounded linear functionals on $X$, which equal zero (annihilates) on $X_1$; the subspace $\left(\tilde{X}_1^*\right)^{\perp}$ is a set of all bounded linear functionals on $\tilde{X}_1^*$, which equal zero (annihilates) on $\tilde{X}_1^*$.***

Suppose $x \in X_1$ so $\langle x, x^* \rangle = 0$ for all $x^* \in X_1^{\perp}$, thus, we have $x \in \left(X_1^{\perp}\right)^{\perp}$. The subset $\left(X_1^{\perp}\right)^{\perp}$ is closed in the strong norm so $\left(X_1^{\perp}\right)^{\perp} \supset clos[X_1]$. Let $x \neq clos[X_1]$ according to the Hahn-Banach theorem gives $x^* \in X_1^{\perp}$, $\langle x, x^* \rangle \neq 0$, and so $x \notin \left(X_1^{\perp}\right)^{\perp}$ thus $\left(X_1^{\perp}\right)^{\perp} = clos[X_1]$. Analogously, the closure of the $\tilde{X}_1^*$ coincides with $\left(\left(\tilde{X}_1^*\right)^{\perp}\right)^{\perp}$.

## 2 Orthogonality in the Banach space

Let us consider the finite rank operators.

**Theorem 1.** ***Assume operator $A$ has finite rank and there is a pair of sets $\{u_i\} \subset Y$ and $\{v_i^*\} \subset X^*$ such that***

$$A = \sum_{i=1}^{k} u_i \otimes v_i^*$$

***then***

$$A^* = \sum_{i=1}^{k} v_i^* \otimes u_i \text{ ,}$$

***where $v_i^* \in X^*$ and $u_i \in Y$ for all $1 \le i \le k$.***

Proof. Assume $y \in Y$ , $v^* \in X^*$, and $x \in X$ , $\tilde{y}^* \in Y^*$, then we write

$$\left\langle \left(y \otimes v^*\right)(x), \tilde{y}^* \right\rangle_Y = \left\langle v^*(x) y, \tilde{y}^* \right\rangle_Y =$$
$$= v^*(x) \left\langle y, \tilde{y}^* \right\rangle_Y .$$

Now, for all $y \in Y$ let us denote map $f_y : Y^* \to C$ such that $f_y\left(y^*\right) = y^*(y) \in Y^{**} \overset{reflex}{=} Y$, so, we have

$$v^*(x), \tilde{y}^*(y) = \left\langle x, \tilde{y}^*(y) v^* \right\rangle_X =$$
$$= \left\langle x, f_y\left(y^*\right) v^* \right\rangle_X = \left\langle x, \left(f_y \otimes v^*\right)\left(y^*\right) \right\rangle_X .$$

Since $\left(f_y \otimes v^*\right)$ is a linear bounded map from $Y^*$ to $X^*$, we have

$$\left(y \otimes v^*\right)^* = f_y \otimes v^* = y \otimes v^* .$$

Thus, adjoints of $u_i$ and $v_i^*$ belong to $v_i^* \otimes u_i$ so an adjoint to the sum is the sum of the adjoint to the terms. Theorem 1 has been proven.

**Theorem 2.** ***Suppose $X_1$ is a subspace of a Banach space $X$. Then we have the following statements.***

***1. Each $v^* \in \left(clos[X_1]\right)^*$ extends to a functional on whole $x^* \in X^*$. We denote isomorphism $\sigma : v^* \to x^* + \left(clos[X_1]\right)^{\perp}$. Then isomorphism $\sigma$ is an isometry from $\left(clos[X_1]\right)^*$ to $X^* / \left(clos[X_1]\right)^{\perp}$.***

***2. Let us denote the map $\pi : X \to X / clos[X_1]$. Then for each $w^* \in \left(X / clos[X_1]\right)^*$, the isomorphism $\tau$ defined as***

$$\tau w^* = w^* \pi$$

***is an isometry from $\left(X/clos\left[X_1\right]\right)^*$ to $\left(clos\left[X_1\right]\right)^{\perp}$.***

**Proof.**

First, the restriction of any $x^* \in X^*$ to $clos\left[X_1\right]$ belongs $\left(clos\left[X_1\right]\right)^*$ so the range of $\sigma$ is whole $X^*/\left(clos\left[X_1\right]\right)^{\perp}$. For any fixed $v^* \in \left(clos\left[X_1\right]\right)^*$ and $x^* \in X^*$ that extends $v^*$, we have an estimation $\left\|v^*\right\| \le \left\|x^*\right\|$, the infimum of such $\left\|x^*\right\|$ equals $\left\|x^* + clos\left[X_1\right]\right\|$. Therefore, we obtain $\left\|v^*\right\| \le \left\|\sigma v^*\right\| \le \left\|x^*\right\|$, however, according to the Hahn-Banach theorem if $x^*$ is an extension of $v^*$ then $\left\|x^*\right\| = \left\|v^*\right\|$ and so we have that $\left\|\sigma v^*\right\| = \left\|v^*\right\|$ is proven.

Second, let $x \in X$ and $w^* \in \left(X/clos\left[X_1\right]\right)^*$ so $\pi x \in X/clos\left[X_1\right]$, and since map $x$ to $w^*\pi x$ vanishes on $x \in clos\left[X_1\right]$, we have $\tau w^* \in \left(clos\left[X_1\right]\right)^{\perp}$. We denote the null space of $x^*$ by $N$, this null space contains $clos\left[X_1\right]$, so there is linear functional $f$ on $X/clos\left[X_1\right]$ such that $f\pi = x^*$, so that $N(f) = \pi(N) \subseteq X/clos\left[X_1\right]$. Thus, we have $f \in \left(X/clos\left[X_1\right]\right)^*$, therefore $\tau f = f\pi = x^*$ so $rang[\tau] = \left(clos\left[X_1\right]\right)^{\perp}$. For any open ball $B$ and any $w^* \in \left(X/clos\left[X_1\right]\right)^*$, we have

$$\left\|\tau w^*\right\| = \left\|w^*\pi\right\| = \sup\left\{\left|\left\langle \pi x, w^*\right\rangle\right| : x \in B\right\} =$$
$$= \sup\left\{\left|\left\langle w, w^*\right\rangle\right| : v \in \pi B\right\} = \left\|w^*\right\|$$

since $\tau w^* = w^*\pi$. The theorem has been proven.

So, if we are given a set $X_1$ in a Banach space $X$, then we have that there are sets: $clos\left[X_1\right] \subset X$ and $\left(clos\left[X_1\right]\right)^* \subset X^*$, $\left(clos\left[X_1\right]\right)^{\perp} \subset X^*$ and $\left(X/clos\left[X_1\right]\right)^* \subset X^*$, and $X^*/\left(clos\left[X_1\right]\right)^{\perp} \subset X^*$, which satisfies the relationships

$$\left(clos\left[X_1\right]\right)^* = X^*/\left(clos\left[X_1\right]\right)^{\perp} \subset X^*$$

and

$$\left(clos\left[X_1\right]\right)^{\perp} = \left(X/clos\left[X_1\right]\right)^* \subset X^*,$$

these set equalities must be understood in the sense of isometric isomorphisms.

**Theorem 3.** ***Let $X$ be a reflexive Banach space. Let us denote $\{e_1, e_2, ..., e_n, ...\}$ and $\{e_1^*, e_2^*, ..., e_n^*, ...\}$ the orthonormal bases in $X$ and $X^*$, respectively. Let sets $E = \vee_{i=1,...,n} e_i$ and $E^* = \vee_{i=1,...,n} e_i^*$ be the closure of the spans $\{e_1, e_2, ..., e_n\}$ and $\{e_1^*, e_2^*, ..., e_n^*\}$ respectively. Let map $P: X \to X$ be a projection in sense $P\,P = P$ and let this projection satisfies the following condition: $P(X) = E$.***

***Then, we have***

$$P = \sum_{k=1,...,n} e_k \otimes e_k$$

***and***

$$P(u) = \sum_{k=1,...,n} \left\langle u, e_k^*\right\rangle e_k$$

***for any $u \in X$.***

Proof. Set $E$ is an n-dimensional closed subset of $X$. Assume $Q(u) = \sum_{k=1,...,n} \left\langle u, e_k^*\right\rangle e_k$ since the definition $\left\langle e_i, e_j^*\right\rangle = \delta_{ij}$, for each $1 \le j \le n$, we have

$$\left\langle Q(u), e_j^*\right\rangle = \sum_{k=1,...,n} \left\langle u, e_k^*\right\rangle \left\langle e_k, e_j^*\right\rangle = \left\langle u, e_j^*\right\rangle$$

so

$$\left\langle u - Q(u), e_j^*\right\rangle = 0,$$

Thus, we conclude $u - Q(u) \perp E^*$ so $u - Q(u) \in \left(E^*\right)^{\perp}$. Let us take $u = u_1 + u_2$, where $u_1 \in E$ and $u_2 \in \left(E^*\right)^{\perp}$. Then we have $u_1 + u_2 - Q(u) \in \left(E^*\right)^{\perp}$ thus $u_1 - Q(u) \in \left(E^*\right)^{\perp}$. On another hand, since $u$ is a finite sum of elements of $E$ so $u_1 - Q(u) \in E$, and we have $u_1 - Q(u) = 0$. Thus, and conclude $Q(u) = P(u)$ since $P(u) = u_1$. The theorem is proven.

**Notation.** ***Let us denote the set of all compact operators from $X$ to $Y$ by $B_C(X, Y)$.***

***And let us denote the set of all finite rank operators as $B_{Ran}(X,Y)$.***

**Theorem (of approximation) 4.** ***Set $B_{Ran}(X,Y)$ is dense in $B_C(X,Y)$. That means if operator $A \in B_C(X,Y)$ then there is a sequence $\{A_n\}$ of operators $A_n \in B_{Ran}(X,Y)$ such that $A_n \to A$.***

Proof. The set $U = \left[A(x)\right]$ is closed and separable. If the set $U$ has a finite dimension then the operator $A$ is a finite rank operator. Or else, assume $\{e_1, e_2, ..., e_n, ...\}$ and $\{e_1^*, e_2^*, ..., e_n^*, ...\}$ are the orthonormal bases of $U$ in $Y$ and $Y^*$, respectively. Let operator $P_n : Y \to Y$ be defined as in the previous theorem for $\vee_{i=1,...,n} e_i$ then $P_n \in B_{Ran}(X,Y)$. Thus, we construct the approximating sequence as $A_n = P_n A \in B_{Ran}(X,Y)$.

For $u \in U$ we get $A(u) = \sum_{k=1,...,...} \left\langle A(u), e_k^* \right\rangle e_k$. So, operators $A_n$ can be written in the form

$$A_n(u) = P_n A(u) = \sum_{k=1,...,n} \left\langle A(u), e_k^* \right\rangle e_k$$

so, we have $\left\| A_n(u) - A(u) \right\| \overset{n\to\infty}{\to} 0$.

Let $B$ be a close ball radius one. Since the operator $A$ is totally bounded we have that for any $\varepsilon > 0$ there is $m$ and $u_1, ..., u_m \in X$ such that $A(B) \subseteq \bigcup_j^m B_\varepsilon\left(A(u_j)\right)$. If $u \in B$ then there is $1 \le j \le m$ such that $A(u) \in B_\varepsilon\left(A(u_j)\right)$ and $\left\| A(u) - A(u_j) \right\| \le \varepsilon$. Since $\|P_n\| \le 1$ we have

$$\begin{aligned}
&\left\| A(u) - A_n(u) \right\| \le \left\| A(u) - A(u_j) \right\| + \\
&+ \left\| A(u_j) - A_n(u_j) \right\| + \left\| P_n\left(A(u_j) - A(u)\right) \right\| \le \\
&\le 2\left\| A(u) - A(u_j) \right\| + \left\| A(u_j) - A_n(u_j) \right\| \le \\
&\le 2\varepsilon + \left\| A(u_j) - A_n(u_j) \right\|.
\end{aligned}$$

So, for each $j$, there is $n(j)$ such that $\left\| A(u_j) - A_n(u_j) \right\| \le \varepsilon$ since $\left\| A(u_j) - A_n(u_j) \right\| \to 0$. For $n \ge \max_{1 \le j \le m} n(j)$ and $u \in B$, we have

$$\left\| A(u) - A_n(u) \right\| \le 3\varepsilon.$$

By definition of operator norm, we write

$$\left\| A - A_n \right\| = \sup_{\|u\| \le 1} \left\| A(u) - A_n(u) \right\|$$

thus, for all, $n \ge \max_{1 \le j \le m} n(j)$ there is the estimation

$$\left\| A - A_n \right\| \le 3\varepsilon,$$

which means that $A_n \to A$. The approximation theorem has been proven.

# 3 Trace operators

Let set $E$ be an orthonormal basis in reflexive Banach space as described above. Elements of the set $E$ can be presented in the form $u = \sum_{e \in E,\ e^* \in E^*} \left\langle u, e^* \right\rangle e$ for all $u \in E$.

Let $X$ and $Y$ be two reflexive Banach spaces and let map $A : X \to Y$ be a bounded linear operator, and $A^*$ be its adjoint operator $A^* : Y^* \to X^*$. Let us denote the set of eigenvalues of $A$ by $\{\lambda_i : i \in I\}$, then the set of eigenvalues of $A^*$ is $\{\bar{\lambda}_i : i \in I\}$. Assume there is an orthonormal basis $E_X$ of $X$ each element of which is an eigenvector of $A$ associated with eigenvalues of $A$.

Let $e_Y \in Y$ and $e_X{}^* \in X^*$ then there is the linear map $e_Y \,\square\, e_X{}^* : X \to Y$.

**Definition 6.** ***We introduce a sequence of the linear maps $X \to Y$ defined by $\sum_i \lambda_i (e_Y)_i \,\square\, (e_X{}^*)_i$, and we introduce an adjoint sequence of the linear adjoint maps $Y^* \to X^*$ defined by $\sum_i \bar{\lambda}_i (e_X{}^*)_i \,\square\, (e_Y)_i$.***

The norms of $A$ and $A^*$ equal to $\|A\| = \sup_{i \in I} \{|\lambda_i|\}$ and $\|A^*\| = \sup_{i \in I} \{|\bar{\lambda}_i|\}$, respectively. That is consistent with the previous result $\|A\| = \|A^*\|$, which holds for the arbitrary operator $A : X \to Y$ and arbitrary pair of the Banach spaces.

Also, we remark that $\left\langle A, A^* \right\rangle = \|A\| \|A^*\|$, however, it does not necessarily mean that operators are self-adjoint i.e. $A \ne A^*$ also possible.

Let us clarify the definition of trace class in the reflexive Banach space.

**Definition 7.** ***Let*** $\{(e_X)_i : i \in I\}$ ***and*** $\{(e_Y)_i : i \in I\}$ ***be orthogonal bases for reflexive Banach spaces*** $X$ ***and*** $Y$***, respectively. We say that linear operator*** $A \in B(X,Y)$ ***is trace class if***

$$\sum_{i\in I} \left\langle A\left((e_X)_i\right), \left(e_Y{}^*\right)_i \right\rangle < \infty .$$

***The trace of the linear operator*** $A \in B(X,Y)$ ***is***

$$Trac(A) = Tr(A) = \sum_{i\in I} \left\langle A\left((e_X)_i\right), \left(e_Y{}^*\right)_i \right\rangle < \infty .$$

In the definition of the trace class, the sum is independent of the bases of the Banach spaces.

***The map*** $Trac : B_C(X,Y) \to \mathbb{C}$ ***is a linear function and if the operator*** $A$ ***is positive then*** $Tr(A)$ ***is real and not negative. If the operator*** $A$ ***is positive then there are orthogonal bases*** $\{(e_X)_i : i \in I\}$ ***and*** $\{(e_Y)_i : i \in I\}$ ***for Banach spaces*** $X$ ***and*** $Y$ ***such that*** $A$ ***can be presented as***

$$A = \sum_i \left\langle A\left((e_X)_i\right), \left(e_Y{}^*\right)_i \right\rangle (e_Y)_i \otimes \left(e_X{}^*\right)_i$$

***and*** $A^*$ ***in the form***

$$A^* = \sum_i \left\langle (e_X)_i, A^*\left(\left(e_Y{}^*\right)_i\right) \right\rangle \left(e_X{}^*\right)_i \otimes (e_Y)_i .$$

**Theorem 5.** ***Let*** $\{(e_X)_n\}$ ***and*** $\{(e_Y)_n\}$ ***be a basis in*** $X$ ***and*** $Y$***, and let*** $A : X \to Y$ ***be a linear operator. If*** $A\left((e_X)_n\right) = \lambda_n (e_Y)_n$ ***for all*** $n$***, then*** $A \in B_C(X,Y)$ ***if and only if*** $\lambda_n \xrightarrow{n\to\infty} 0$***.***

Proof. Let $A \in B_C(X,Y)$ then $A^* \in B_C(Y^*, X^*)$. Let $\lambda_n = \lambda\left(A\left((e_X)_n\right)\right)$ is an eigenvalue corresponding to $(e_X)_n$. Let $(P_X)_n$ be the projection onto $\vee_{i=1,\dots,n} e_i$ as defined above. So, we have $\left((P_X)_n\right)^* A^* \to A^*$, and we define $A_n = A - A(P_X)_n$. Since $\|A_n\| = \|A_n{}^*\|$ we get

$$\|A_n\| = \|A_n{}^*\| = \left\|\left(A - A(P_X)_n\right)^*\right\| = $$
$$= \left\|A^* - \left((P_X)_n\right)^* A^*\right\| \to 0,$$

where $A^* : Y^* \to X^*$. Since $\|A_n\| \to 0$ we have $\lim_{n\to\infty} \sup_{j>n} |\lambda_n| = 0$.

On another hand, since $\lambda_n \xrightarrow{n\to\infty} 0$ all absolute values of the eigenvalues are bounded so $A \in B(X,Y)$. For natural numbers $k \le n$, we have $A\left((P_X)_n\left((e_X)_j\right)\right) = A\left((e_X)_j\right) = \lambda_j (e_Y)_j$ and for $k > n$ $A\left((P_X)_n\left((e_X)_j\right)\right) = 0$, thus $A(P_X)_n \in B_{Ran}(X,Y)$. Let us denote $A_n = A - A(P_X)_n$ so $\|A_n\| = \sup_{j>n} |\lambda_n|$, thus $\lim_{n\to\infty} \|A_n\| = 0$.

We have obtained that the sequence $A(P_X)_n$ converges to $A$ as $n \to \infty$, here all $A(P_X)_n \in B_{Ran}(X,Y)$, therefore $A \in B_C(X,Y)$. The theorem is proven.

## 4 Examples

Let us consider several examples.

1. Let $X = R^2$ and $Y = R^3$, linear operator $A : R^2 \to R^3$, $y = Ax$ is defined by $y_i = \sum_{k=1,2} a_{ik} x_k$, $i = 1, 2, 3$, where $a_{ik}$ are components of the [2, 3] - matrix $[A_{ik}]$, $rank[A_{ik}] = 2$. Then $A^* : R^3 \to R^2$ is defined as $x = A^{\mathbf{T}} y$ or $x_i = \sum_{k=1,2,3} a_{ki} y_k$, $i = 1, 2, 3$. We can define the operator $A^* A : R^2 \to R^2$ by the multiplication of the matrices $A^{\mathbf{T}} \times A$, the product is a square [2, 2] matrix; the operator $AA^* : R^3 \to R^3$ analogously is given by $A \times A^{\mathbf{T}}$ whose product is a square [3, 3] – matrix. The operator $A : R^2 \to R^3$ is embedding $R^2$ in $R^3$, and the operator $A^* : R^3 \to R^2$ is the projection $R^3$ on $R^2$.

2. In our next example, let $X = L^p(R)$ and $Y = L^q(R)$, $p + q = pq$, $p > 1$, then the linear map $A : L^p(R) \to L^q(R)$ is defined as an integral operator with the singular kernel $K(t,s)$, which

satisfies certain conditions (for instant Gaussian distribution), in the form

$$y(t) = Ax(t) = \int K(t,s)\,x(s)\,ds .$$

The adjoint operator $A^{*}:\ L^{p}(R) \to L^{q}(R)$ is also defined by a singular integral with the transpose kernel $K^{*}(t,s)$.

3. The most important example is the case, when $X = Y = H$, where $H$ is a Hilbert space. Let the map $A: H \to H$ be a compact operator, then $A^{*}A = AA^{*}$ if and only if there is an orthonormal basis of space each vector of which is its eigenvector. So, there is a basis $\{e_i\}$ so that $Ae_i = \lambda_i e_i$, $i \in I$, then $A^{*}e_i = \bar{\lambda}_i e_i$, so

$$AA^{*}e_k = A\sum_i \bar{\lambda}_i \langle e_k, e_i \rangle e_i =$$

$$= A\left(\bar{\lambda}_k e_k\right) = \bar{\lambda}_k \lambda_k e_k = A^{*}Ae_k .$$

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