Generalization of the Fourier calculus and Wigner function

MYKOLA YAREMENKO
Department of Partial Differential Equations,
The National Technical University of Ukraine,
"Igor Sikorsky Kyiv Polytechnic Institute", Kyiv, UKRAINE

Abstract: - In this paper, we consider l_p-periodical functions $pcs(m\,\theta)$ and $psn(m\,\theta)$, which are defined on the curve given by the equation: $|x|^p+|y|^p=1, \quad p>1$ on R^2 as functions of its length. Considering $pcs(m\,\theta)$ and $psn(m\,\theta)$ as an independent functional system, we construct the theory similar to Fourier analysis with the proper weights. For these weights, we establish an analogous of the Riemannian theorem. The adjoint representations are introduced and dual theory is developed. These Fourier representations can be used for approximation of the oscillation processes.

Keywords: - General periodic function, Fourier analysis, p-circle, adjoint, p-Laplacian, linear approximation, spectral theory, oscillation.

Received: August 26, 2021. Revised: April 17, 2022. Accepted: May 18, 2022. Published: July 3, 2022.

Introduction

A curved line given by the equation $|x|^p+|y|^p=1$ on R^2-plane is called a p-curve and denoted by Cp. Let us denote the length of p-curve by l_p. We introduce a pair of C^1-smooth functions $pcs(\theta)$ and $psn(\theta)$ of the real argument $\theta\in[0,l_p]$ defined as

$$pcs(\theta)=x \quad \text{for all} \quad \theta\in R \tag{1}$$

and

$$psn(\theta)=y \quad \text{for all} \quad \theta\in R, \tag{2}$$

where coordinates x and y belongs to p-curve, i.e. bound by the equation $|x|^p+|y|^p=1$, so that

$$psn(0)=pcs\left(\frac{l_p}{4}\right)=0 \text{ and}$$

$$pcs(0)=psn\left(\frac{l_p}{4}\right)=1, \text{ and}$$

$$|psn(\theta)|^p+|pcs(\theta)|^p=1 \quad \text{for all} \quad \theta\in R. \tag{3}$$

These functions satisfy the integral identity

$$psn(\theta)\,pcs(\theta)= \\ =\int\left(\left(pcs(\theta)\right)^p-\left(psn(\theta)\right)^p\right)d\theta. \tag{4}$$

p-Fourier transform

Assume $f\in L^p[0,l_p]$ and let us write a Fourier-type series with appropriate weights on the interval $[0,l_p]$ as

$$f(x)= \\ =a_0+\sum_{m=1,2,\dots}\left(a_m pcs(mx)+b_m psn(mx)\right), \tag{5}$$

with some real coefficients $a_0,a_1,b_1,..,a_m,b_m,...$.

By usual means. integrating the identity (3) over the period l_p, we obtain

$$\int_0^{l_p}|pcs(\theta)|^p\,d\theta=\int_0^{l_p}|psn(\theta)|^p\,d\theta=\frac{l_p}{2} \tag{6}$$

and

$$a_0 = \frac{1}{l_p}\int_0^{l_p} f(x)\,dx. \quad (7)$$

Next, we have

$$a_m = \frac{2}{l_p}\int_0^{l_p} f(x)\,pcs(mx)\left|pcs(mx)\right|^{p-2} dx \quad (8)$$

and

$$b_m = \frac{2}{l_p}\int_0^{l_p} f(x)\,psn(mx)\left|psn(mx)\right|^{p-2} dx. \quad (9)$$

Thus, we obtain the mapping of the functions $f \in L^p\left[0, l_p\right]$ in the set of the infinite series according to the formula

$$f(x) = \frac{1}{l_p}\int_0^{l_p} f(x)\,dx + \frac{2}{l_p}\sum_{m=1,2,\dots}\left(\begin{matrix}\int_0^{l_p} f(y)\,pcs(my)\left|pcs(my)\right|^{p-2} pcs(mx) + \\ +\int_0^{l_p} f(y)\,psn(my)\left|psn(my)\right|^{p-2} psn(mx)\end{matrix}\right) dy. \quad (10)$$

Statement (analogous Riemannian theorem) 1. Assuming g is an integrable function over an arbitrary interval $[a, b] \subset R$ then

$$\lim_{m\to\infty}\int_a^b g(x)\,psn(mx)\left|psn(mx)\right|^{p-2} dx = 0 \quad (11)$$

and

$$\lim_{m\to\infty}\int_a^b g(x)\,pcs(mx)\left|pcs(mx)\right|^{p-2} dx = 0. \quad (12)$$

Theorem (adjoint) 2. Let g be an integrable function over an arbitrary interval $[a, b] \subset R$ then there are

$$\lim_{m\to\infty}\int_a^b g(x)\,psn(mx)\,dx = 0 \quad (13)$$

and

$$\lim_{m\to\infty}\int_a^b g(x)\,pcs(mx)\,dx = 0. \quad (14)$$

Adjoint series

Assume $f \in L^p$ then $f\left|f\right|^{p-2} \in L^{\frac{p}{p-1}}$ and we can write

$$f(x)\left|f(x)\right|^{p-2} = \tilde{a}_0 + \sum_{m=1,2,\dots}\left(\begin{matrix}\tilde{a}_m pcs(mx)\left|pcs(mx)\right|^{p-2} + \\ \tilde{b}_m psn(mx)\left|psn(mx)\right|^{p-2}\end{matrix}\right), \quad (15)$$

where $\tilde{a}_0, \tilde{a}_1, \tilde{b}_1, \dots, \tilde{a}_m, \tilde{b}_m, \dots$ defined as follows

$$\tilde{a}_0 = \frac{1}{l_p}\int_0^{l_p} f(x)\left|f(x)\right|^{p-2} dx, \quad (16)$$

$$\tilde{a}_m = \frac{2}{l_p}\int_0^{l_p} f(x)\left|f(x)\right|^{p-2} pcs(mx)\,dx \quad (17)$$

and

$$\tilde{b}_m = \frac{2}{l_p}\int_0^{l_p} f(x)\left|f(x)\right|^{p-2} psn(mx)\,dx. \quad (18)$$

The morphism from the real line to the complex plane $Epp : R \to Cp$

We introduce a function $Epp : R \to Cp$, which maps from the real line to the p-curve on the complex plane as follows

$$Epp(i\theta) = pcs(\theta) + i\,psn(\theta), \quad \theta \in R \quad (19)$$

and dual function

$$Epq(i\theta) = pcs(\theta) + i\,psn(\theta), \quad \theta \in R, \quad p = q, \quad (20)$$

assume that p is renaming q. The function $Epp : R \to Cp$ is a surjective morphism of the topological groups from the real line R to the p-curve Cp and covering the space of the p-curve Cp. In case $p = 2$, the function Epp is a classical exponent on the complex plane of the imaginary argument.

From formula (19), we have

$$pcs(\theta) = \frac{1}{2}\left(Epp(i\theta) + Epp(-i\theta)\right), \quad \theta \in R$$

and

$$psn(\theta) = \frac{1}{2i}\left(Epp(i\theta) - Epp(-i\theta)\right), \quad \theta \in R.$$

We introduce an integral transformation Tp of a function $f \in L^p \cap L^q$ in the form

$$ {}^{p}\hat{f}(\lambda) = \int_{-\infty}^{\infty} Epp\left(-l_p i\lambda \cdot x\right) f(x)\, dx = Tp(f)(\lambda) \quad (21) $$

where l_p is a length of the p-curve Cp.

This integral transformation Tp is a linear mapping relative to the function f and in case $p=2$ coincides with the Fourier transformation.

If $p=2$ then the integral transformation of function g

$$ \int_{-\infty}^{\infty} Epp\left(l_p i\lambda \cdot x\right) g(\lambda)\, d\lambda = Rp(g)(x) \quad (22) $$

coincides with the inverse Fourier transform, in the general case it is not necessarily true since the dual structure does not coincide with the natural complex structure, the inverse transform is not always given by formula (22).

We define the inverses integral transformation Tp^{-1} of a function ${}^{p}\hat{f}(\lambda)$ as

$$ f(x) = Tp^{-1}\left({}^{p}\hat{f}\right)(x) \quad (23) $$

for all transforms ${}^{p}\hat{f}(\lambda)$.

So, we introduce two types of mappings: the first is an analog of the Fourier transform Tp and its inverse Tp^{-1}, second is an analog of the inverse Fourier transform Rp and we can easily define its inverse Rp^{-1}. These morphisms do not have the structure of the group except for $p=2$.

Generalization of the Wigner function

Let functions $\psi \in L^p\left(R^n\right)$ and $\varphi \in L^q\left(R^n\right)$ then we introduce a general Wigner function $W_\eta(\psi,\varphi)(x,p)$ as any quasi-probability distribution, which satisfies the following conditions:

1. $\int_{R^n} W_\eta(\psi,\varphi)(x,p)\, dp = \psi(x)\bar{\varphi}(x)$;
2. $\int_{R^n} W_\eta(\psi,\varphi)(x,p)\, dx = Tp\left(\psi(p)\right)\overline{Tp}\left(\varphi(p)\right)$.

As a consequence of the first condition, we have

$$ \int_{R^{2n}} W_\eta(\psi,\varphi)(x,p)\, dp dx = \left\langle \psi(x)\bar{\varphi}(x)\right\rangle_x . $$

For a pair of functions $\psi \in L^p\left(R^n\right)$ and $\varphi \in L^q\left(R^n\right)$ such that $\langle \psi \mid \varphi\rangle \neq 0$, we define a density ρ in the point (x,p) by

$$ \rho_{\psi,\varphi}(x,p) = \overline{\rho_{\psi,\varphi}(x,p)} = \frac{W_\eta(\psi,\varphi)(x,p)}{\langle \psi \mid \varphi\rangle} . $$

The probability density function is a homogeneous function of degree one so that $\rho_{\lambda\psi,\lambda\varphi}(x,p) = \rho_{\psi,\varphi}(x,p)$ for all complex $\lambda \neq 0$.

Let us introduce the generalization of the Weyl quantization by

$$ \left(\Im_\sigma \psi\right)(\lambda) = \int_{R^n} Epp\left(l_p i\, \sigma(\lambda, x)\right)\psi(x) dx , $$

where σ is a symplectic form.

We define an operator

$$ V(\lambda) = Epp\left(-l_p i\, \sigma\left((\lambda, x),(Q,P)\right)\right), $$

where Q is position operators and P is a momentum.

The Weyl quantization $Dp(\psi)(\phi)$ is defined by

$$ Dp(\psi)(\phi) = \left\langle \left(\Im_\sigma \psi\right)(\cdot) V(\cdot)\phi(\cdot)\right\rangle $$

for any test function ϕ.

We estimate $\left\|Dp(\psi)(\phi)\right\| \leq \left\|\Im_\sigma \psi\right\|_p \|\phi\|$.

Similarly to the classical case, the new Weyl quantization is a linear mapping so that

$$ Dp(\alpha\psi + \beta\varphi) = \alpha Dp(\psi) + \beta Dp(\varphi) $$

holds for all complex numbers α, β.

Definition. *The Schwartz space is a space of all functions such that*

$$S\left(R^n\right)=\begin{Bmatrix}\psi\in C^{\infty}\left(R^n\right) & : \\ \sup_{x\in R^n}\left|x^a\partial_x^{\alpha}\psi\left(x\right)\right|<\infty & \forall\alpha,a\in N^n\bigcup\{0\}\end{Bmatrix}.$$

Now, let us consider a case when $Epp=Exp$. The exponent function satisfies the characteristic identity $Exp\left(a+b\right)=Exp\left(a\right)Exp\left(b\right)$ so the Weyl product has the property

$$Dp\left(\psi\#\varphi\right)=Dp\left(\psi\right)Dp\left(\varphi\right)$$

for some function ψ,φ.

The symbol # denotes a non-commutative product (often called Weyl product) so that $Dp\left(\psi\#\varphi\right)=Dp\left(\psi\right)\cdot Dp\left(\varphi\right)$ for some functions.

Let us assume K_A and K_B are kernels for the integral operators A and B respectively. So, we have

$$Dp\left(Dp^{-1}\left(A\right)\phi\right)\left(x\right)=\int_{R^{2n}}\exp\left(-2\pi i\left(z-x\right)p\right)\varepsilon^n\times W_{\eta}\left(K_A\right)\left(\frac{1}{2}\left(x+z,\varepsilon p\right)\right)\phi\left(z\right)dpdz=$$
$$\varepsilon^n\int_{R^{3n}}\exp\left(-2\pi i\left(z-x+y\right)p\right)\times K_A\left(\frac{1}{2}\left(z+x+y\right),\frac{1}{2}\left(z+x-y\right)\right)\phi\left(z\right)dpdzdy,$$

we take $Dp\left(\psi\right)=A$ then $\psi=Dp^{-1}\left(A\right)$ and calculate

$$K_A\left(x+\frac{\varepsilon}{2}z,x-\frac{\varepsilon}{2}z\right)=\varepsilon^{-n}\left(F^{-1}\psi\right)\left(x,z\right),$$

thus

$$Dp^{-1}\left(Dp\left(\psi\right)\right)\left(x,p\right)=\psi\left(x,p\right).$$

Generally speaking, the product $K_A \square K_B\in S\left(R^n\times R^n\right)$ does not commute. So, we obtain the following lemma.

Lemma 1. ***Let*** K_A ***be a kernel of an operator*** $A\in BL\left(L^2\left(R^n\right),L^2\left(R^n\right)\right)$. ***Then the mapping*** Dp^{-1} ***is an inverse to Weyl quantization so that*** $Dp^{-1}A=\varepsilon^nW\left(K_A\right)$ ***and*** $A=Dp\left(\varepsilon^nW\left(K_A\right)\right)$***; the Weyl kernel is given by***

$$K_{\psi}=\int_{R^n}\exp\left(-2\pi i\left(z-x\right)p\right)\psi\left(\frac{1}{2}\left(x+z,\varepsilon p\right)\right)dp$$
$$=\varepsilon^n\left(F\left(\psi\right)\right)\left(\frac{1}{2}\left(x+z,\frac{z-x}{\varepsilon}\right)\right),$$

then

$$Dp^{-1}\left(Dp\left(\psi\right)\right)\left(x,p\right)=$$
$$\varepsilon^n\left(W\left(K_{\psi}\right)\right)\left(x,p\right)=\psi\left(x,p\right)$$

holds for $\psi\in L^2\left(R^n\right)$.

Lemma 2. ***Let*** K_A ***and*** K_B ***be integral kernels of the operators*** A ***and*** B ***respectively. Then the product*** $\left(K_A \square K_B\right)\left(x,z\right)=\left\langle K_A\left(x,\cdot\right)K_B\left(\cdot,z\right)\right\rangle$ ***is correctly defined and is a kernel of the operator; in other words*** $\bullet:S\left(R^n\times R^n\right)\times S\left(R^n\times R^n\right)\to S\left(R^n\times R^n\right)$.

Proof. Let us denote the multi-indices by $a,\alpha,b,\beta\in N_0^n$ then we estimate

$$\left|x^az^b\partial_x^{\alpha}\partial_z^{\beta}\left(K_A \square K_B\right)\left(x,z\right)\right|=$$
$$\left|x^az^b\partial_x^{\alpha}\partial_z^{\beta}\left\langle K_A\left(x,\cdot\right)K_B\left(\cdot,z\right)\right\rangle\right|\le$$
$$\le\left\langle\left|x^az^b\partial_x^{\alpha}\partial_z^{\beta}K_A\left(x,\cdot\right)K_B\left(\cdot,z\right)\right|\right\rangle=$$
$$\left\|x^az^b\partial_x^{\alpha}\partial_z^{\beta}K_A\left(x,\cdot\right)K_B\left(\cdot,z\right)\right\|_{L^1}\le$$
$$\le Const1\sup_{\cdot\in R^n}\left|x^az^b\partial_x^{\alpha}\partial_z^{\beta}K_A\left(x,\cdot\right)K_B\left(\cdot,z\right)\right|+$$
$$+Const2\max_{|c|=2n}\sup_{\cdot\in R^n}\left|x^az^b\partial_x^{\alpha}\partial_z^{\beta}K_A\left(x,\cdot\right)K_B\left(\cdot,z\right)\right|\le$$
$$\le Const1\left\|x^az^b\partial_x^{\alpha}\partial_z^{\beta}K_A\left(x,\cdot\right)K_B\left(\cdot,z\right)\right\|_{\mathbf{00}}+$$
$$+Const2\max_{|c|=2n}\left\|x^az^b\partial_x^{\alpha}\partial_z^{\beta}K_A\left(x,\cdot\right)K_B\left(\cdot,z\right)\right\|_{c\mathbf{0}}.$$

Next, we exchange the order of the supremum and integration and obtain

$$\sup_{x,\, z\in R^n}\left\langle \left| x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x,\cdot) K_B(\cdot, z)\right|\right\rangle \le$$

$$\left\langle \sup_{x,\, z\in R^n}\left| x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x,\cdot) K_B(\cdot, z)\right|\right\rangle =$$

$$= \left\| \sup_{x,\, z\in R^n}\left| x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x,\cdot) K_B(\cdot, z)\right|\right\|_{L^1}$$

so, we have

$$\left\| \sup_{x,\, z\in R^n}\left| x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x,\cdot) K_B(\cdot, z)\right|\right\|_{L^1} \le$$

$$\le C(1) \sup_{x,\, z\in R^n} \sup_{\cdot\in R^n} \left| x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x,\cdot) K_B(\cdot, z)\right| +$$

$$C(2) \max_{|c|=2n} \sup_{x,\, z\in R^n} \sup_{\cdot\in R^n} \left| x^a z^b \partial_x^\alpha \partial_z^\beta K_A(x,\cdot) K_B(\cdot, z)\right| < \infty,$$

thus, we obtain $K_A \square K_B \in S(R^n \times R^n)$.

For the Weyl system, we can formulate the following Weyl quantization theorem.

Theorem. ***Let functions*** $\psi, \varphi \in S(R^{2n})$ ***then the function*** $\psi \# \varphi \in S(R^{2n})$ ***and such that satisfies the equality***

$$Dp(\psi \# \varphi) = Dp(\psi) Dp(\varphi),$$

where

$$(\psi \# \varphi)(x, p) =$$

$$\left\langle \left\langle \begin{matrix} \exp(2\pi i \sigma((x, p), (z+\tilde{z}, \eta+\tilde{\eta}))) \times \\ \exp\left(2\pi \frac{i\varepsilon}{2} \sigma((z,\eta),(\tilde{z},\tilde{\eta}))\right) \times \\ (F_\sigma \psi)(z,\eta)(F_\sigma \varphi)(\tilde{z},\tilde{\eta}) \end{matrix} \right\rangle_{(z,\eta)} \right\rangle_{(\tilde{z},\tilde{\eta})} =$$

$$\left\langle \left\langle \begin{matrix} \exp\left(\begin{matrix} 2\pi \frac{2i}{\varepsilon} \times \\ \sigma\left(\begin{matrix} (x, p)-(z,\eta),(x, p)- \\ (\tilde{z},\tilde{\eta}) \end{matrix}\right) \end{matrix} \right) \times \\ \psi(z,\eta)\varphi(\tilde{z},\tilde{\eta}) \end{matrix} \right\rangle \right\rangle.$$

Proof. Assume $\psi, \varphi \in S(R^{2n})$ and employ the definition of Dp, we have

$$(Dp(\psi) Dp(\varphi)) =$$

$$= \left\langle \left\langle \begin{matrix} (F_\sigma \psi)(z,\eta)(F_\sigma \varphi)(\tilde{z},\tilde{\eta}) \times \\ W(z,\eta) W(\tilde{z},\tilde{\eta}) \end{matrix} \right\rangle_{(z,\eta)} \right\rangle_{(\tilde{z},\tilde{\eta})} =$$

$$\left\langle \left\langle \begin{matrix} \exp\left(-2\pi \frac{2i}{\varepsilon} \sigma\left(\begin{matrix} (z,\eta),(\tilde{z},\tilde{\eta})- \\ (z,\eta) \end{matrix}\right)\right) \times \\ (F_\sigma \psi)(z,\eta)(F_\sigma \varphi)\left(\begin{matrix} (z,\eta)- \\ (\tilde{z},\tilde{\eta}) \end{matrix}\right) W(\tilde{z},\tilde{\eta}) \end{matrix} \right\rangle_{(z,\eta)} \right\rangle_{(\tilde{z},\tilde{\eta})}.$$

Now, we are going to establish that $\psi \# \varphi \in S(R^{2n})$

$$(\psi \# \varphi)(x, p) =$$

$$= F_\sigma \left(\left\langle \begin{matrix} \exp\left(2\pi \frac{2i}{\varepsilon} \sigma\left(\begin{matrix} (z,\eta),\cdot- \\ (z,\eta) \end{matrix}\right)\right) \times \\ (F_\sigma \psi)(z,\eta)(F_\sigma \varphi)(\cdot-(z,\eta)) \end{matrix} \right\rangle_{(z,\eta)} \right)(x, p) =$$

$$\left\langle \left\langle \begin{matrix} \exp(2\pi i \sigma((x, p),(z,\eta))) \times \\ \exp\left(2\pi \frac{i\varepsilon}{2} \sigma((z,\eta),(\tilde{z},\tilde{\eta}))\right) \times \\ (F_\sigma \psi)(z,\eta)(F_\sigma \varphi)(\tilde{z},\tilde{\eta}) \end{matrix} \right\rangle_{(z,\eta)} \right\rangle_{(\tilde{z},\tilde{\eta})} =$$

$$= \left\langle \left\langle \begin{matrix} \exp\left(2\pi \frac{2i}{\varepsilon} \sigma\left(\begin{matrix} (x, p),(\tilde{z},\tilde{\eta})+ \\ (z,\eta) \end{matrix}\right)\right) \times \\ \exp\left(2\pi \frac{2i}{\varepsilon} \sigma((\tilde{z},\tilde{\eta}),(z,\eta))\right) \times \\ (F_\sigma \psi)(z,\eta)(F_\sigma \varphi)(\tilde{z},\tilde{\eta}) \end{matrix} \right\rangle_{(z,\eta)} \right\rangle_{(\tilde{z},\tilde{\eta})}$$

so $\psi \# \varphi$ belongs $S(R^{2n})$.

Let us denote K_ψ and K_φ kernels, which belong to $S(R^{2n})$, then we have

$$\left(Dp(\psi)Dp(\varphi)\phi\right)(x)= =\left\langle\left(K_{\psi}\bullet K_{\varphi}\right)(x,\cdot)\phi(\cdot)\right\rangle\langle\ \rangle_{z}= =\left\langle\left\langle K_{\psi}K_{\varphi}(\cdot,z)\phi(z)\right\rangle\right\rangle_{z}=Dp(\psi\#\varphi)(x).$$

Next, using the properties of the exponential function, we have

$$(\psi\#\varphi)(x,p)= =\int_{R^{8n}}\left(\exp\left(2\pi i\sigma\left((x,p),(z,\eta)+(y,\varsigma)\right)\right)\times\right. \exp\left(2\pi\frac{i\varepsilon}{2}\sigma\left((z,\eta),(y,\varsigma)\right)\right)\times \exp\left(2\pi\frac{i\varepsilon}{2}\sigma\left((z,\eta),(\tilde{z},\tilde{\eta})\right)\right)\times \exp\left(2\pi\frac{i\varepsilon}{2}\sigma\left((y,\varsigma),(\tilde{y},\tilde{\varsigma})\right)\right)\times \left.\psi(\tilde{z},\tilde{\eta})\varphi(\tilde{y},\tilde{\varsigma})\right)dzd\eta dyd\varsigma d\tilde{z}d\tilde{\eta}d\tilde{y}d\tilde{\varsigma}= =\int_{R^{4n}}\left(\exp\left(2\pi i\sigma\left((z,\eta),(\tilde{z},\tilde{\eta})-(x,p)\right)\right)\times\right. \left.\psi(\tilde{z},\tilde{\eta})\varphi\left((x,p)+\frac{\varepsilon}{2}(z,\eta)\right)\right)dzd\eta d\tilde{z}d\tilde{\eta}.$$

By changing variables $(y,\varsigma)=(x,p)+\frac{\varepsilon}{2}(z,\eta)$, we are completing the proof of the theorem.

From semigroup properties of exponential function follows: let a be a symbol of $S\left(R^{2n}\right)$ then the Weyl operator is given by

$$\hat{A}\psi(x)= \left(\frac{1}{2\pi\eta}\right)^{n}\left\langle\begin{matrix}a\left(\frac{1}{2}(x+z),p\right)\times\\ \exp\left(\frac{i}{\eta}p\cdot(x-z)\right)\psi(z)\end{matrix}\right\rangle_{(z,p)},$$

the kernel of the Weyl operator A is

$$K_{\hat{A}}(x,y)= \left(\frac{1}{2\pi\eta}\right)^{n}\left\langle\exp\left(\frac{i}{\eta}p\cdot(x-y)\right)a\left(\frac{1}{2}(x+y),p\right)\right\rangle_{p},$$

and the symbol is written as

$$a(x,p)= \left\langle\exp\left(-\frac{i}{\eta}p\cdot z\right)K_{\hat{A}}\left(x+\frac{1}{2}z,x-\frac{1}{2}z\right)\right\rangle_{z}.$$

These formulae are circular via to the semigroup properties.

Since

$$\hat{T}_{R}\left(x_{0},p_{0}\right)\left(\psi(x)\right)= \exp\left(2\frac{i}{\eta}p_{0}\cdot\left(x-x_{0}\right)\right)\psi\left(2x_{0}-x\right) \tag{24}$$

the Weyl operator can be written in the form

$$\hat{A}\psi(x)= \left(\frac{1}{2\pi\eta}\right)^{n}\left\langle a(z,p)\hat{T}_{R}(z,p)\left(\psi(x)\right)\right\rangle_{(z,p)}. \tag{25}$$

Statement. *The Weyl operator extends to the continuous operator* $\hat{A}:S'\left(R^{n}\right)\to S'\left(R^{n}\right)$**.**

Indeed, Since $a\in S\left(R^{2n}\right)$ the function $a(z,p)x^{\alpha}\partial_{x}^{\alpha}\hat{T}_{R}(z,p)\psi\in S\left(R^{2n}\right)$ for all functions $\psi\in S\left(R^{n}\right)$ and all multi-indices $\alpha\in N^{n}$ therefore $\left|x^{\alpha}\partial_{x}^{\alpha}\hat{A}\psi(x)\right|<\infty$.

Weyl established that correspondence between symbols a and Weyl operators A is one-to-one and linear, unit symbol corresponds to the identity operator on $S'\left(R^{n}\right)$. Thus the set of all Weyl operators coincides with the set of all symbols on $S\left(R^{n}\oplus R^{n}\right)$. The Weyl operators are pseudo-differential operators with rapidly decreasing kernels.

Since the Weyl operator can be rewritten as

$$\hat{A}\psi(x) = \left(\frac{1}{2\pi\eta}\right)^n \left\langle \begin{matrix} a\left(\frac{1}{2}(x+z),\ p\right)\times \\ \exp\left(\frac{i}{\eta} p\cdot(x-z)\right)\psi(z) \end{matrix} \right\rangle_{(z,\ p)}, \quad (26)$$

so that the kernel of the Weyl operator A can be calculated by the formula

$$K_{\hat{A}}(x, y) = \left(\frac{1}{2\pi\eta}\right)^n \left\langle \exp\left(\frac{i}{\eta} p\cdot(x-y)\right) a\left(\frac{1}{2}(x+y),\ p\right)\right\rangle_p, \quad (27)$$

then, the symbol can be represented as

$$a(x, p) = \left\langle \exp\left(-\frac{i}{\eta} p\cdot z\right) K_{\hat{A}}\left(x+\frac{1}{2}z,\ x-\frac{1}{2}z\right)\right\rangle_z. \quad (28)$$

The last three formulae are circular.

Theorem 4. Let $\hat{A}\overset{Weyl}{\leftrightarrow} a$ ***be the Weyl*** **correspondence** ***then***

1. ***for*** $a \in S(R^n \oplus R^n)$ ***it is necessary and sufficient*** $K_{\hat{A}}(x, y) \in S(R^n \oplus R^n)$ and $\hat{A}(\psi(x)) = \langle K_{\hat{A}}(x, z)\psi(z)\rangle_z$***;***
2. ***the map*** $a \mapsto \hat{A}$ extends to an isomorphism $S'(R^n \oplus R^n) \to L(S(R^n), S'(R^n))$, where $L(S(R^n), S'(R^n))$ ***is the space of continuous linear operators from*** $S(R^n)$ ***to*** $S'(R^n)$.

Proof. The theorem follows from the Schwartz kernel theorem.

Theorem 5. ***Let the Weyl operator*** $\hat{A}$ ***corresponds to the symbol*** $a \in L^r(R^n \oplus R^n),\ \ 1 \le r < 2$ ***so*** $\hat{A}\overset{Weyl}{\leftrightarrow} a$***, then there is a constant*** $Const(r)$ ***such that the inequality***

$$\left\|\hat{A}\psi\right\|_{L^2(R^n)} \le Const(r)\|\psi\|_{L^2(R^n)}\|a\|_{L^r(R^{2n})} \quad (29)$$

holds for all $\psi \in L^2(R^n)$***.***

From this theorem follows that for all symbols $a \in L^2(R^n \oplus R^n)$ corresponding Weyl operators are L^2-bounded. However, there are examples of the symbols $a \in L^r(R^n \oplus R^n),\ \ 2 < r$ on which L^2 boundness is ruined so that Weyl operators $\hat{A}$ are not L^2- bounded for these symbols $a \in L^r(R^n \oplus R^n),\ \ 2 < r$.

The complete analysis of L^2- regularity for Weyl operators can be made in terms of the Calderon -Zygmund theory.

Theorem 6. ***Let*** $\hat{A}$ ***be trace-class Weyl operator on*** $L^2(R^n)$ ***corresponded to symbol*** $a \in L^r(R^n \oplus R^n),\ \ 1 \le r < 2$***. Then for*** $\hat{A} \ge 0$ ***it is necessary and sufficient that***

$$F_\sigma a(x, p) = \left\langle \exp\left(i\sigma\left((x, p), (\tilde{x}, \tilde{p})\right)\right) a(\tilde{x}, \tilde{p})\right\rangle_{(\tilde{x},\ \tilde{p})}$$

is continuous and such that the matrix with entries

$$\exp\left(-\frac{i}{2}\eta\sigma\left(\left(x_j, p_j\right), \left(x_k, p_k\right)\right)\right)\times$$
$$F_\sigma a\left(\left(x_j, p_j\right)-\left(x_k, p_k\right)\right)$$

is positive semidefinite for all possible sets of $\left(x_1, p_1\right), \left(x_2, p_2\right), ..., \left(x_N, p_N\right) \in \left(R^{2n}\right)^N$.

References:

[1] Chambolle A., Conti S. and Iurlano F.: Approximation of functions with small jump sets and existence of strong minimizers of Griffith's energy. J. Math. Pures Appl., 128/9 (2019), 119–139.
[2] Chambolle A. and Crismale V.: A density result in GSBDp with applications to the approximation of brittle fracture energies. Arch. Rational Mech. Anal., 232 (2019), 1329–1378.
[3] Conti S., Focardi M., and Iurlano F.: Existence of strong minimizers for the Griffith static fracture model in dimension two. Ann. Inst. H. Poincar´e Anal. Non Lin´eaire, 36 (2019), 455–474.
[4] Conti S., Focardi M., and Iurlano F.: Approximation of fracture energies with p-growth via piecewise affine finite elements. ESAIM Control Optim. Calc. Var., 25 (2019), paper no. 34
[5] Crismale V. and Friedrich M.: Equilibrium configurations for epitaxially strained films and material voids in three-dimensional linear elasticity. Arch. Rational Mech. Anal., 237 (2020), 1041–1098.
[6] El-Nabulsi R.A.: Fractional action cosmology with variable order parameter. Int. J. Theor. Phys. 2017, 56, 1159.
[7] Guo D. and Chu W.: Summation formulae involving multiple Harmonic numbers, Appl. Anal. Discrete Math. 15(1) (2021), 201–212.
[8] Kim D. and Simsek Y.: A new family of zeta type function involving Hurwitz zeta function and the alternating Hurwitz zeta function, Mathematics 9(3) (2021), 233.
[9] Krantz S.G.: Handbook of complex variables, Springer Science, New York (1999).
[10] Ma M. and Lim D.: Degenerate Derangement Polynomials and Numbers, Fractal Fract. 5(3) (2021), 59.
[11] Murphy G.M.: Ordinary Differential Equations and Their Solutions; Dover Publication, Inc.: New York, NY, USA, 2011.
[12] Mathai A.M. and Haubold H.J.: Special Functions for Applied Scientists; Springer: New York, NY, USA, 2008.
[13] Prodanov D.: Regularized Integral Representations of the Reciprocal Gamma Function. Fractal Fract 75 2019, 3, 1.
[14] Reynolds R. and Stauffer A.: Definite Integral of Arctangent and Polylogarithmic Functions Expressed as 77 a Series. Mathematics 2019, 7, 1099.
[15] Reynolds R. and Stauffer A.: A definite integral involving the logarithmic function in terms of the Lerch 79 Function. Mathematics 2019, 7, 1148.
[16] Reynolds R. and Stauffer A.: Derivation of Logarithmic and Logarithmic Hyperbolic Tangent Integrals 81 Expressed in Terms of Special Functions. Mathematics 2020, 8, 687.
[17] Reynolds R. and Stauffer A.: Definite integrals involving the product of logarithmic functions and logarithm 83 of square root functions expressed in terms of special functions., AIMS Mathematics, 5, 2020.
[18] Saha A. and Talukdar B.: Inverse variational problem for nonstandard Lagrangians. Rep. Math. Phys. 2014, 73, 299–309.
[19] Udwadia F.E. and Cho H.: Lagrangians for damped linear multi-degree-of-freedom systems. J. Appl. Mech. 2013, 80, 041023.
[20] Usman T., Khan N., Saif M., and Choi J.: A Unified Family of Apostol-Bernoulli Based Poly-Daehee Polynomials, Montes Taurus J. Pure Appl. Math. 3(3) (2021), 1–11.