Analysis of the Doppler Effect Based on the Full Maxwell Equations

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Abstract: - In the previous paper, a modification of Maxwell's equations was proposed, from which formula for Doppler effect follows. However, as was noted later, the equations proposed do not have symmetry with respect to the transformation B→-E, E→B, which the original Maxwell equations have and which was discovered by Heaviside in 1893. The equations proposed in present paper have this symmetry. The obtained equations are analyzed for several physical situations.

Key-Words: - Maxwell Equations; Doppler Effect; Symmetry of Equations

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1 Introduction to the Problem

In the paper [1], the formula for the Doppler effect was given, which describes the dependence of the radiation frequency recorded by the observer, depending on the angle between the direction to the source and the direction of movement of the source. As shown in our work [2], the expression for the Doppler effect can be written as

$$\left(\frac{\Omega}{k}\frac{\boldsymbol{k}}{k}-\boldsymbol{v}\right)^2=c^2, \qquad (1)$$

which is just a record of the cosine theorem for the difference between the velocity vectors of the wave front and the source. Here $\mathbf{k}$ is the wave vector, Ω - frequency, $\boldsymbol{v}$ - vector of movement of the source of light, c - the speed of light.

In the same place, a modification of Maxwell's equations was proposed, from which formula (1) follows. However, as was noted later, the equations proposed in [2] do not have symmetry with respect to the transformation

$$\boldsymbol{B}\to-\boldsymbol{E}, \quad \boldsymbol{E}\to\boldsymbol{B},$$

which the original Maxwell equations have and discovered by Heaviside in 1893 (see [3]). The equations proposed below have this kind symmetry as shown in the present paper.

2 Mathematical model

2.1 The Modified Maxwell Equations

Consider the modified Maxwell equations of the following form:

$$\begin{aligned}&\frac{d\boldsymbol{B}}{dt}+[\boldsymbol{v}\times\boldsymbol{\nabla}]\times\boldsymbol{B}+\boldsymbol{\nabla}\times\boldsymbol{E}=0,\\&\frac{d\boldsymbol{E}}{dt}+[\boldsymbol{v}\times\boldsymbol{\nabla}]\times\boldsymbol{E}-\boldsymbol{\nabla}\times\boldsymbol{B}+\boldsymbol{j}=0.\end{aligned} \qquad (2)$$

Here and further, the Heaviside system of units is used everywhere, according to which ħ = c = 1.

Here the time derivatives are total, i.e., $d/dt=\partial/\partial t+(v\cdot\nabla)$, where v - constant speed of the source.

2.2 Fourier Expansion for the Modified Maxwell Equations

Since the equations are linear with constant coefficients, we use the Fourier expansion to solve them:

$$\boldsymbol{B}=\boldsymbol{b}\cdot e^{i(-\omega t+\boldsymbol{k}\cdot\boldsymbol{x})},$$

$$\boldsymbol{E} = \boldsymbol{e} \cdot e^{i(-\omega t + \boldsymbol{k}\cdot\boldsymbol{x})},$$

$$\boldsymbol{j} = \boldsymbol{j_k} \cdot e^{i(-\omega t + \boldsymbol{k}\cdot\boldsymbol{x})}.$$

In this representation referred to in the literature as the impulse representation, system (2) has the form:

$$\begin{aligned} &s\boldsymbol{b} + [\boldsymbol{v} \times \boldsymbol{k}] \times \boldsymbol{b} + \boldsymbol{k} \times \boldsymbol{e} = 0, \\ &s\boldsymbol{e} + [\boldsymbol{v} \times \boldsymbol{k}] \times \boldsymbol{e} - \boldsymbol{k} \times \boldsymbol{b} = i\boldsymbol{j_k} \end{aligned} \qquad (3)$$

where $s \equiv -\omega + \boldsymbol{v} \cdot \boldsymbol{k}$.

The determinant of the system (3) is

$$\det = s^2(s^2 - k^2 + [\boldsymbol{v} \times \boldsymbol{k}]^2)^2. \qquad (4)$$

By substitution

$$\begin{aligned} &\boldsymbol{b} = i\boldsymbol{k} \times (\boldsymbol{A} - \boldsymbol{v}\Phi) + i\boldsymbol{k}\Psi, \\ &\boldsymbol{e} = -is(\boldsymbol{A} - \boldsymbol{v}\Phi) - i\boldsymbol{k}(\Phi - \boldsymbol{v} \cdot \boldsymbol{A}) + i[\boldsymbol{v} \times \boldsymbol{k}]\Psi, \end{aligned} \qquad (5)$$

where are:

$\boldsymbol{A}$ – vector,
Φ – scalar
Ψ – pseudoscalar

potentials, we can get the following conditions for the potentials:

$$\boldsymbol{A} = -\frac{\boldsymbol{j}}{(s^2 - k^2 + [\boldsymbol{v} \times \boldsymbol{k}]^2)}, \qquad (6)$$

$$s\Phi + \boldsymbol{k} \cdot \boldsymbol{A} = 0, \qquad (7)$$

$$s\Psi + [\boldsymbol{v} \times \boldsymbol{k}] \cdot \boldsymbol{A} = 0. \qquad (8)$$

2.3 The Modified Maxwell Equations in Coordinate Form

The above is presented in the coordinate form as follows:

$$\boldsymbol{b} = \boldsymbol{\nabla} \times (\boldsymbol{A} - \boldsymbol{v}\Phi) + \boldsymbol{\nabla}\Psi, \qquad (9)$$

$$\boldsymbol{e} = -\frac{d}{dt}(\boldsymbol{A} - \boldsymbol{v}\Phi) - \boldsymbol{\nabla}(\Phi - \boldsymbol{v} \cdot \boldsymbol{A}) + [\boldsymbol{v} \times \boldsymbol{\nabla}]\Psi,$$

where are

$$\left(\frac{d^2}{dt^2} - \boldsymbol{\nabla}^2 + [\boldsymbol{v} \times \boldsymbol{\nabla}]^2\right)\boldsymbol{A} = \boldsymbol{j}, \qquad (10)$$

$$\frac{d}{dt}\Phi + \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0, \qquad (11)$$

$$\frac{d}{dt}\Psi + \boldsymbol{v} \times \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0. \qquad (12)$$

Applying the operator $\left(\frac{d^2}{dt^2} - \boldsymbol{\nabla}^2 + [\boldsymbol{v} \times \boldsymbol{\nabla}]^2\right)$ to the calibration condition $\frac{d}{dt}\Phi + \boldsymbol{\nabla} \cdot \boldsymbol{A} = 0$, yields

$$\frac{d}{dt}\left(\frac{d^2}{dt^2} - \boldsymbol{\nabla}^2 + [\boldsymbol{v} \times \boldsymbol{\nabla}]^2\right)\Phi + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0.$$

And using the continuity equation $\frac{d}{dt}\rho + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$, we obtain an expression for the charge density:

$$\rho \equiv \left(\frac{d^2}{dt^2} - \boldsymbol{\nabla}^2 + [\boldsymbol{v} \times \boldsymbol{\nabla}]^2\right)\Phi. \qquad (13)$$

It follows from the first formula (9) that the divergence of the magnetic field is not equal to zero in motion, while at rest it is still zero:

$$\boldsymbol{\nabla} \cdot \boldsymbol{b} = \nabla^2\Psi, \qquad (14)$$

therefore, the lines of force of a magnetic field source moving at a constant speed are not closed circles in classical electrodynamics.

2.4 The Galilean Transformations

Applying the formulae (5) – (8) yields:

$$\begin{aligned} &i\boldsymbol{k} \cdot (\boldsymbol{e} - \boldsymbol{v} \times \boldsymbol{b}) = -(s^2 - k^2 + [\boldsymbol{v} \times \boldsymbol{k}]^2)\Phi, \\ &i\boldsymbol{k} \cdot (\boldsymbol{b} + \boldsymbol{v} \times \boldsymbol{e}) = +(s^2 - k^2 + [\boldsymbol{v} \times \boldsymbol{k}]^2)\Psi, \end{aligned} \qquad (15)$$

or in coordinate form

$$\boldsymbol{\nabla} \cdot (\boldsymbol{e} - \boldsymbol{v} \times \boldsymbol{b}) = \rho. \qquad (16)$$

The expressions in (9):

$$\begin{aligned} &\boldsymbol{A}' = \boldsymbol{A} - \boldsymbol{v}\Phi \\ &\Phi' = \Phi - \boldsymbol{v} \cdot \boldsymbol{A} \end{aligned} \qquad (17)$$

obviously mean the transition to a moving frame of reference (*the Galilean transformations*).

Thus, the scalar potential Φ and vector potential $\boldsymbol{A}$ create a pair in these transformations. And a pseudoscalar potential Ψ does not take part in this as seen from direct calculation: the expression $\boldsymbol{v} \cdot \boldsymbol{\nabla} \times \boldsymbol{A}$ is invariant under transformation (17).

2.5 The Inverse Galilean Transformations

The inverse Galilean transformations have the form:

$$A = \frac{A' + v\Phi'}{1 - v^2}$$
$$\Phi = \frac{\Phi' + v \cdot A'}{1 - v^2} \qquad (18)$$

Unlike the Lorentz transformations used in relativistic electrodynamics, formulas (17) and (18) are asymmetric, since in formulas (17) the medium in which the wave propagates is at rest, and in (18) it moves relative to the coordinate system with a speed of $-\boldsymbol{v}$.

3 Change of Variables

The equations (15) suggest the idea to make a change of variables:

$$\boldsymbol{\varepsilon} = \boldsymbol{e} - \boldsymbol{v} \times \boldsymbol{b}$$
$$\boldsymbol{\beta} = \boldsymbol{b} + \boldsymbol{v} \times \boldsymbol{e} \qquad (19)$$

In such variables the system (2) transforms to:

$$\frac{d\boldsymbol{\varepsilon}}{dt} - \boldsymbol{\nabla} \times \left(\boldsymbol{\beta} - \boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{\beta})\right) + \boldsymbol{j} = 0,$$
$$\frac{d\boldsymbol{\beta}}{dt} + \boldsymbol{\nabla} \times \left(\boldsymbol{\varepsilon} - \boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{\varepsilon})\right) + [\boldsymbol{v} \times \boldsymbol{j}] = 0. \qquad (20)$$

From the continuity equation $\frac{d}{dt}\rho + \boldsymbol{\nabla} \cdot \boldsymbol{j} = 0$ follows

$$\boldsymbol{\nabla} \cdot \boldsymbol{\varepsilon} = \rho, \qquad (21)$$

4 Energy-Momentum Tensor and Dispersion Equation

4.1 Energy-Momentum Tensor

Let us scalarly multiply the first of equations (20) by the vector $\boldsymbol{\varepsilon} - \boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{\varepsilon})$, and the second by the vector $\boldsymbol{\beta} - \boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{\beta})$**,** and add, we get

$$\frac{d}{dt}\frac{\left(\varepsilon^2 - (\boldsymbol{v} \cdot \boldsymbol{\varepsilon})^2 + \beta^2 - (\boldsymbol{v} \cdot \boldsymbol{\beta})^2\right)}{2} + \nabla \cdot \left(\boldsymbol{\varepsilon} \times \boldsymbol{\beta} + \boldsymbol{v} \times \left[\boldsymbol{v} \times [\boldsymbol{\varepsilon} \times \boldsymbol{\beta}]\right]\right) = -(\boldsymbol{\varepsilon} - \boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{\varepsilon}) - [\boldsymbol{v} \times \boldsymbol{\beta}]) \cdot \boldsymbol{j}$$

And then multiply vectorially (on the right) the first of equations (20) by the vector $\boldsymbol{\beta}$, β, and the second (on the left) by the vector $\boldsymbol{\varepsilon}$ and add, we get

$$\frac{d}{dt}[\boldsymbol{\varepsilon} \times \boldsymbol{\beta}] + \boldsymbol{\nabla}\frac{(\beta^2 - (\boldsymbol{v} \cdot \boldsymbol{\beta})^2 + \varepsilon^2 - (\boldsymbol{v} \cdot \boldsymbol{\varepsilon})^2)}{2} - \boldsymbol{\beta} \cdot \boldsymbol{\nabla}\left(\boldsymbol{\beta} - \boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{\beta})\right) - \boldsymbol{\varepsilon} \cdot \boldsymbol{\nabla}\left(\boldsymbol{\varepsilon} - \boldsymbol{v}(\boldsymbol{v} \cdot \boldsymbol{\varepsilon})\right) = [\boldsymbol{\beta} \times \boldsymbol{j}] + \boldsymbol{\varepsilon} \times [\boldsymbol{v} \times \boldsymbol{j}].$$

4.2 Dispersion Equation

Dispersion equation for the system (20) $det = 0$ has the following form:

$$s^2(s^2 - k^2 + [\boldsymbol{v} \times \boldsymbol{k}]^2)^2 = 0. \qquad (22)$$

The roots of this equation are:

$s_0 = 0$ (longitudinal mode),

$s_\pm = \pm\sqrt{k^2 - [\boldsymbol{v} \times \boldsymbol{k}]^2}$ (transversal modes).

Recalling the definition of s, we have:

$$\omega_0 = \boldsymbol{v} \cdot \boldsymbol{k} \qquad (23)$$

and

$$\Omega_\pm = \boldsymbol{v} \cdot \boldsymbol{k} \pm \sqrt{k^2 - [\boldsymbol{v} \times \boldsymbol{k}]^2}. \qquad (24)$$

Denoting by θ the angle between the vectors $\boldsymbol{v}$ and $\boldsymbol{k}$, we obtain the last expression in the form

$$\Omega_\pm = k \cdot v \cdot cos\theta \pm k \cdot \sqrt{1 - v^2 \cdot sin\theta^2},$$

or

$$V_\pm = v \cdot cos\theta \pm \sqrt{1 - v^2 \cdot sin\theta^2}. \qquad (25)$$

Here $V_\pm \equiv \frac{\Omega_\pm}{k}$ – the amplitude of the phase speed for transversal wave. This expression (25) was given in [1] ($c = 1$ was accepted in (25)).

Formula (25) is actually a record of the cosine theorem for the difference of vectors. Expressions for the group velocity and the delayed Green's function are given in [2].

5 Conclusion

From the results presented above follows that the non-relativistic explanation of the Doppler effect, based on the concept of a continuous medium in which elastic - longitudinal and transverse - oscillations propagate, may explain some more about the Doppler effect, as well as about the other interesting features.

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