An Approach to Optimal Filtering of Time-Variant Systems via Finite Measurements

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Abstract: Fast optimal estimates are often required in control and signal processing. In this paper, we discuss an approach to optimal finite impulse response (OFIR) filtering for discrete time-variant systems using finite measurements. The mean square error is minimized to obtain the batch OFIR algorithm which requires measurements on an a finite horizon of N points. Fast iterative algorithm is found using recursions. It is shown that each recursion has a predictor/corrector Kalman filter (KF)-like format with special initial conditions. In this sense, the KF is considered as a special case of the proposed iterative OFIR filtering algorithm when N approaches infinity for known initial conditions. It has been confirmed by simulation that the iterative form of the OFIR filter operates much faster than the batch form.

Key–Words: Optimal FIR filter, iterative algorithm, time-variant system, Kalman filter

1 Introduction

The real-time state model $\mathbf{x}_k = \mathbf{A}_k\mathbf{x}_{k-1} + \mathbf{B}_k\mathbf{w}_k$, where $\mathbf{A}_k$ and $\mathbf{B}_k$ are the system matrices, $\mathbf{x}_k$ is the system state, and $\mathbf{w}_k$ is the noise vector, is commonly used in signal processing when prediction is not an issue [1, 2]. Employing this model, many filtering algorithms have been designed and employed. Among them, the finite impulse response (FIR) is a method using finite most recent measurements to compute the system states [3–5]. Due to this interesting filtering structure, many useful advantages are achieved such as better robustness against temporary modeling uncertainties and higher immunity against errors in the noise statistics. This has generated profound research studies in optimal FIR filtering [6–21].

For example, the unbiased FIR (UFIR) filter and smoother were proposed in [3] for polynomial systems. In [9], a p-shift UFIR filter (UFIR) was derived as a special case of the optimal FIR (OFIR) filter. Here, the unbiasedness was checked a posteriori and the solution thus belongs to CU. Soon after, the UFIR filter [9] was extended to time-variant systems [12, 15]. For nonlinear models, an extended UFIR filter was proposed in [17] and unified forms for FIR filtering and smoothing were discussed in [18]. The method of determining the optimal horizon in the UFIR filter was discussed in [19], and the optimal unbiased FIR filter was proposed in [20], where the resulting method is proposed by minimizing the mean square error (MSE) constrained by the unbiasedness condition. An important advantage of UFIR filtering is that the noise statistics are not required and noise reduction is provided by averaging. Therefore, the estimation horizon for the UFIR filter must be optimal.

It has been shown that the OFIR filter is full-horizon, to mean that the estimation errors decrease with the increase in the estimation horizon. This quite useful property implies that one does not need to compute an optimal horizon: a relatively large one can ensure a good performance. Since the OFIR approach uses the noise statistics, its gain is more complex than that of the UFIR filter which ignores noises. Up to now, fast iterative OFIR filtering has been developed only for time-invariant models [21]. It still remains an open problem for time-varying ones.

In this paper, the batch and iterative forms of the OFIR filter are derived for discrete time-variant system model with Gaussian white noise. Compared to the infinite impulse response (IIR) filters, the proposed method inherits advantages of FIR structures and is more robust against temporary modeling uncertainties. On the other hand, compared to the UFIR filter given in [9], the OFIR filter does not strictly requires the optimal horizon. This is because the optimal performance of the OFIR filter is guaranteed by large averaging horizons.

2 State-Space Model and Preliminaries

In the state space, we consider a linear discrete time-variant system described by

$$\begin{aligned}\mathbf{x}_k &= \mathbf{A}_k\mathbf{x}_{k-1} + \mathbf{B}_k\mathbf{w}_k\,, &(1)\\ \mathbf{y}_k &= \mathbf{C}_k\mathbf{x}_k + \mathbf{D}_k\mathbf{v}_k\,, &(2)\end{aligned}$$

where $\mathbf{x}_k \in \mathbb{R}^n$ is the state vector, $\mathbf{A}_k \in \mathbb{R}^{n\times n}$, $\mathbf{B}_k \in \mathbb{R}^{n\times u}$, $\mathbf{C}_k \in \mathbb{R}^{q\times n}$ and $\mathbf{D}_k \in \mathbb{R}^{q\times v}$ are system matrices, and $\mathbf{y}_k \in \mathbb{R}^q$ is the measurement. The process noise $\mathbf{w}_k \in \mathbb{R}^u$ and measurement noise $\mathbf{v}_k \in \mathbb{R}^v$ are zero mean Gaussian, $E\{\mathbf{w}_k\} = \mathbf{0}$ and $E\{\mathbf{v}_k\} = \mathbf{0}$, mutually uncorrelated and have known covariances, $\mathbf{R} = E\{\mathbf{w}_k\mathbf{w}_k^T\}$ and $\mathbf{Q} = E\{\mathbf{v}_k\mathbf{v}_k^T\}$.

The FIR estimator can be expressed as a linear combination of finite samples of measurements on the horizon of N of points in the form of

$$\hat{\mathbf{x}}_{k|k} = \mathbf{K}_k\mathbf{Y}_{k,l}, \qquad (3)$$

where $l = k - N + 1$ is the starting point of the horizon, N is the horizon length, $\hat{\mathbf{x}}_{k|k}$ is the estimate, $\mathbf{Y}_{k,l}$ is a vector measurements collecting on a horizon $[l,k]$, and $\mathbf{K}_k$ is the filter gain determined by a given performance criterion.

Compared with the IIR structure, a distinct feature of FIR estimator is that N most recent measurements are employed at each time step, while only one most recent measurement is used in IIR (Kalman) form. This leads to $\mathcal{O}(N)$ complexity. However, some good properties such as the BIBO stability and better robustness are achieved. We formulate the problem as follows: Given the model, (1) and (2), we would like to derive the batch form and iterative form of the OFIR filter in minimum mean square error (MMSE) sense, and provide a comparison with the UFIR filter ignoring noise statistics and KF.

3 OFIR Filtering Algorithm

In order to derive the OFIR filter on a horizon of N past measurements from l to k, we represent (1) and (2) in a batch form as

$$\begin{aligned}\mathbf{X}_{k,l} &= \mathbf{A}_{k,l}\mathbf{x}_l + \mathbf{B}_{k,l}\mathbf{W}_{k,l}\,, &(4)\\ \mathbf{Y}_{k,l} &= \mathbf{C}_{k,l}\mathbf{x}_l + \mathbf{H}_{k,l}\mathbf{W}_{k,l} + \mathbf{D}_{k,l}\mathbf{V}_{k,l}\,. &(5)\end{aligned}$$

Here, $\mathbf{X}_{k,l} \in \mathbb{R}^{Nn}$, $\mathbf{Y}_{k,l} \in \mathbb{R}^{Nq}$, $\mathbf{W}_{k,l} \in \mathbb{R}^{Nu}$ and $\mathbf{V}_{k,l} \in \mathbb{R}^{Nv}$ are specified as, respectively,

$$\begin{aligned}\mathbf{X}_{k,l} &= \left[\mathbf{x}_k^T\mathbf{x}_{k-1}^T\cdots\mathbf{x}_l^T\right]^T, &(6)\\ \mathbf{Y}_{k,l} &= \left[\mathbf{y}_k^T\mathbf{y}_{k-1}^T\cdots\mathbf{y}_l^T\right]^T, &(7)\\ \mathbf{W}_{k,l} &= \left[\mathbf{w}_k^T\mathbf{w}_{k-1}^T\cdots\mathbf{w}_l^T\right]^T, &(8)\\ \mathbf{V}_{k,l} &= \left[\mathbf{v}_k^T\mathbf{v}_{k-1}^T\cdots\mathbf{v}_l^T\right]^T. &(9)\end{aligned}$$

The extended model matrix $\mathbf{A}_{k,l} \in \mathbb{R}^{Nn\times n}$, process noise matrix $\mathbf{B}_{k,l} \in \mathbb{R}^{Nn\times Nu}$, observation matrix $\mathbf{C}_{k,l} \in \mathbb{R}^{Nq\times n}$, auxiliary process noise matrix $\mathbf{H}_{k,l} \in \mathbb{R}^{Nq\times Nu}$ and measurement noise matrix $\mathbf{D}_{k,l} \in \mathbb{R}^{Nq\times Nv}$ are all time-variant and dependent on the current time k and the horizon length N. Model (1) and (2) suggests that these matrices can be written as, respectively

$$\mathbf{A}_k^l = [\mathcal{A}_k^{l+1^T}, \mathcal{A}_{k-1}^{l+1^T}, \cdots, \mathcal{A}_{l+1}^{l+1^T}, \mathbf{I}]^T, \qquad (10)$$

$$\mathbf{B}_{k,l} = \begin{bmatrix} \mathbf{B}_k & \mathcal{A}_k^k\mathbf{B}_{k-1} & \cdots & \mathcal{A}_k^{l+2}\mathbf{B}_{l+1} & \mathcal{A}_k^{l+1}\mathbf{B}_l \\ \mathbf{0} & \mathbf{B}_{k-1} & \cdots & \mathcal{A}_{k-1}^{l+2}\mathbf{B}_{l+1} & \mathcal{A}_{k-1}^{l+1}\mathbf{B}_l \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{B}_{l+1} & \mathcal{A}_{l+1}^{l+1}\mathbf{B}_l \\ \mathbf{0} & \mathbf{0} & \cdots & \mathbf{0} & \mathbf{B}_l \end{bmatrix}, \qquad (11)$$

$$\begin{aligned}\mathbf{C}_{k,l} &= \bar{\mathbf{C}}_{k,l}\mathbf{A}_{k,l}, &(12)\\ \mathbf{H}_{k,l} &= \bar{\mathbf{C}}_{k,l}\mathbf{B}_{k,l}, &(13)\\ \mathbf{D}_{k,l} &= \mathrm{diag}\left(\mathbf{D}_k\mathbf{D}_{k-1}\cdots\mathbf{D}_l\right), &(14)\end{aligned}$$

with

$$\begin{aligned}\mathcal{A}_\psi^\zeta &= \begin{cases} \mathbf{A}_\psi\mathbf{A}_{\psi-1}\cdots\mathbf{A}_\zeta, & \text{if } \psi > \zeta \\ \mathbf{A}_\psi, & \text{if } \psi = \zeta \end{cases}, &(15)\\ \bar{\mathbf{C}}_{k,l} &= \mathrm{diag}\left(\mathbf{C}_k\mathbf{C}_{k-1}\cdots\mathbf{C}_l\right), &(16)\end{aligned}$$

where $\psi \geqslant \zeta$. Note that the state equation specified by (4) and (5) at the initial point l is $\mathbf{x}_l = \mathbf{x}_l + \mathbf{B}_l\mathbf{w}_l$, suggesting that $\mathbf{w}_l$ is zero-valued. That is, the initial state $\mathbf{x}_l$ is required to be known or estimated optimally. In the following, we concentrate our attention to develop the optimal FIR estimator for the above system.

3.1 Batch Computational Form

By combining (5) with (3), we provide

$$\hat{\mathbf{x}}_{k|k} = \mathbf{K}_k\left(\mathbf{C}_{k,l}\mathbf{x}_l + \mathbf{H}_{k,l}\mathbf{W}_{k,l} + \mathbf{D}_{k,l}\mathbf{V}_{k,l}\right). \qquad (17)$$

Now, our objective is to compute the optimal gain $\hat{\mathbf{K}}_k$ to minimize the covariance of estimation error in the minimum MSE sense. In other words, the following cost function must be minimized

$$\hat{\mathbf{K}}_k = \arg\min_{\mathbf{K}_k} E\left\{\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\right)\left(\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}\right)^T\right\} . \tag{18}$$

To compute (18), the orthogonality principle can be employed. Specifically, the optimal filter gain $\hat{\mathbf{K}}_k$ should guarantee the estimation error $\mathbf{x}_k - \hat{\mathbf{x}}_{k|k}$ is uncorrelated with any of the measurements $\mathbf{Y}_{k,l}$, and also to any of the linear combination of these measurements. In this sense, an equivalent way to rewrite (18) is

$$E\left\{\left(\mathbf{x}_k - \hat{\mathbf{K}}_k\mathbf{Y}_{k,l}\right)\left(\mathbf{Y}_{k,l}\right)^T\right\} = \mathbf{0} , \tag{19}$$

in which $\mathbf{x}_k$ can be constructed as

$$\mathbf{x}_k = \mathcal{A}_k^{l+1}\mathbf{x}_l + \bar{\mathbf{B}}_{k,l}\mathbf{W}_{k,l} , \tag{20}$$

where $\bar{\mathbf{B}}_{k,l}$ is the first row vector of $\mathbf{B}_{k,l}$. Substituting (5) and (20) into (19), using the fact that the initial state $\mathbf{x}_l$, systems noise vector $\mathbf{W}_{k,l}$ and measurement noise $\mathbf{V}_{k,l}$ are pairwise independent, and taking the expectation and rearranging the terms, (19) becomes

$$\mathcal{A}_k^{l+1}\mathbf{\Theta}_{x,l}\mathbf{C}_{k,l}^T + \bar{\mathbf{Z}}_{w,k} = \hat{\mathbf{K}}_k\mathbf{Z}_{x+w+v,k} , \tag{21}$$

where auxiliary matrices are

$$\begin{aligned}
\mathbf{\Theta}_l &= E\left\{\mathbf{x}_l\mathbf{x}_l^T\right\} , & (22)\\
\mathbf{Z}_{x,k} &= \mathbf{C}_{k,l}\mathbf{\Theta}_l\mathbf{C}_{k,l}^T , & (23)\\
\mathbf{Z}_{w,k} &= \mathbf{H}_{k,l}E\left[\mathbf{W}_{k,l}\mathbf{W}_{k,l}^T\right]\mathbf{H}_{k,l}^T , & (24)\\
\mathbf{Z}_{v,k} &= \mathbf{D}_{k,l}E\left[\mathbf{V}_{k,l}\mathbf{V}_{k,l}^T\right]\mathbf{D}_{k,l}^T , & (25)\\
\bar{\mathbf{Z}}_{w,k} &= \bar{\mathbf{B}}_{k,l}E\left[\mathbf{W}_{k,l}\mathbf{W}_{k,l}^T\right]\mathbf{H}_{k,l}^T , & (26)\\
\mathbf{Z}_{x+w+v,k} &= \mathbf{Z}_{x,k} + \mathbf{Z}_{w,k} + \mathbf{Z}_{v,k} . & (27)
\end{aligned}$$

Notations $\mathbf{Z}_{x,k}$, $\mathbf{Z}_{w,k}$ and $\mathbf{Z}_{v,k}$ denote the covariances of initial state, systems and measurement noise respectively, which are intuitively symmetric and invertible. Referring to these properties, we multiply both sides of (21) from the right-hand side with $\mathbf{Z}_{x+w+v,k}^{-1}$ and find the optimal gain in the form of

$$\hat{\mathbf{K}}_k = (\mathcal{A}_k^{l+1}\mathbf{\Theta}_l\mathbf{C}_{k,l}^T + \bar{\mathbf{Z}}_{w,k})\mathbf{Z}_{x+w+v,k}^{-1} . \tag{28}$$

Further multiplying $\mathbf{\Theta}_l$ on the right hand of (28) with the identity matrix $(\mathbf{C}_{k,l}^T\mathbf{C}_{k,l})^{-1}\mathbf{C}_{k,l}^T\mathbf{C}_{k,l}$, from the left-hand side, the optimal filter gain $\hat{\mathbf{K}}_k$ can be equivalently rewritten in a more compact way as

$$\hat{\mathbf{K}}_k = \bar{\mathbf{K}}_k\mathbf{Z}_{x,k}\mathbf{Z}_{x+w+v,k}^{-1} + \bar{\mathbf{Z}}_{w,k}\mathbf{Z}_{x+w+v,k}^{-1} , \tag{29}$$

where

$$\bar{\mathbf{K}}_k = \mathcal{A}_k^{l+1}\left(\mathbf{C}_{k,l}^T\mathbf{C}_{k,l}\right)^{-1}\mathbf{C}_{k,l}^T . \tag{30}$$

In fact, $\bar{\mathbf{K}}_k$ is the unbiased filter gain. By substituting (30) into (3) and averaging both sides, one may find out that the unbiasedness constraint $E\left[\hat{\mathbf{x}}_{k|k}\right] = E\left[\mathbf{x}_k\right]$ is guaranteed [16]. In order to compute (29), the covariance of initial state is required to be determined. Toward this end, the following discrete algebraic Riccati equation (DARE) [24] can be developed.

$$\begin{aligned}
&\mathbf{Y}_{k,l}\mathbf{Y}_{k,l}^T\mathbf{Z}_{w+v,k}^{-1}\mathbf{Z}_{x,k} - \mathbf{Z}_{x,k}\mathbf{Z}_{w+v,k}^{-1}\mathbf{Z}_{x,k}\\
&\qquad -2\mathbf{Z}_{x,k} - \mathbf{Z}_{w+v,k} = \mathbf{0}
\end{aligned} \tag{31}$$

where

$$\mathbf{Z}_{w+v,k} = \mathbf{Z}_{w,k} + \mathbf{Z}_{v,k} . \tag{32}$$

4 Iterations

To avoid constructing and computing the vectors and matrices with N-dependent dimensions, the iterative form for the OFIR filter can be summarized as follows.

Theorem 1 *Given the discrete time-variant state space model (1) and (2) with zero mean and mutually independent noise vectors $\mathbf{w}_k$ and $\mathbf{v}_k$ having Gaussian distributions and known covariances, the iterative form for the OFIR filter can be stated by*

$$\begin{aligned}
\mathbf{N}_l &= \mathbf{\Theta}_l + \mathbf{B}_l\mathbf{R}\mathbf{B}_l^T , & (33)\\
\hat{\mathbf{x}}_l &= \mathbf{N}_l\mathbf{C}_l^T\mathbf{Z}_{x+w+v,l}^{-1}\mathbf{y}_l , & (34)\\
\hat{\mathbf{x}}_i &= \mathbf{A}_i\hat{\mathbf{x}}_{i-1} + \mathbf{N}_i\mathbf{C}_i^T\left(\tilde{\mathbf{Q}}_i + \mathbf{C}_i\mathbf{N}_i\mathbf{C}_i^T\right)^{-1} & \\
&\quad \times\left(\mathbf{y}_i - \mathbf{C}_i\mathbf{A}_i\hat{\mathbf{x}}_{i-1}\right) , & (35)
\end{aligned}$$

where i ranges from $l+1$ to k, $\mathbf{N}_i$ is computed recursively by (67), and the initial mean square state $\mathbf{\Theta}_l$ can be obtained by solving (31).

To find an iterative form for (28), taking i as an iterative variable, and employing (10), (12) and (13) and decomposing $\mathbf{C}_{i,l}$, $\mathbf{H}_{i,l}$, and $\mathbf{D}_{i,l}$ as, respectively,

$$\begin{aligned}
\mathbf{C}_{i,l} &= \left[(\mathbf{C}_i\mathcal{A}_i^{l+1})^T\ \mathbf{C}_{i-1,l}^T\right]^T , & (36)\\
\mathbf{H}_{i,l} &= \begin{bmatrix} \mathbf{C}_i\mathbf{B}_i & \mathbf{C}_i\mathbf{A}_i\bar{\mathbf{B}}_{i-1,l} \\ \mathbf{0} & \mathbf{H}_{i-1,l} \end{bmatrix} , & (37)\\
\mathbf{D}_{i,l} &= \begin{bmatrix} \mathbf{D}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{D}_{i-1,l} \end{bmatrix} . & (38)
\end{aligned}$$

Then, we get

$$\mathbf{Z}_{x,i} = \begin{bmatrix} \mathbf{C}_i\mathcal{A}_i^{l+1}\mathbf{\Theta}_l\mathcal{A}_i^{l+1^T}\mathbf{C}_i^T & \mathbf{C}_i\mathcal{A}_i^{l+1}\mathbf{\Theta}_l\mathbf{C}_{i-1,l}^T \\ \mathbf{C}_{i-1,l}\mathbf{\Theta}_l\mathcal{A}_i^{l+1^T}\mathbf{C}_i^T & \mathbf{C}_{i-1,l}\mathbf{\Theta}_l\mathbf{C}_{i-1,l}^T \end{bmatrix} \tag{39}$$

$$\mathbf{Z}_{w,i} = \begin{bmatrix} \mathbf{C}_i\bar{\mathbf{B}}_{i,l}\mathbf{R}_i\bar{\mathbf{B}}_{i,l}^T\mathbf{C}_i^T & \mathbf{C}_i\mathbf{A}_i\bar{\mathbf{Z}}_{w,i-1}^T \\ \bar{\mathbf{Z}}_{w,i-1}^T\mathbf{A}_i^T\mathbf{C}_i^T & \mathbf{Z}_{w,i-1} \end{bmatrix} \quad (40)$$

$$\mathbf{Z}_{v,i} = \begin{bmatrix} \mathbf{D}_i\mathbf{Q}\mathbf{D}_i^T & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_{v,i-1} \end{bmatrix}, \quad (41)$$

where

$$\bar{\mathbf{Z}}_{w,i-1} = \bar{\mathbf{B}}_{i-1,l}\mathbf{R}_{i-1}\mathbf{H}_{i-1,l}^T . \quad (42)$$

By introducing $\mathbf{\Delta}_i$, $\mathbf{F}_i$ and $\mathbf{U}_i$ as, respectively,

$$\mathbf{\Delta}_i \triangleq \begin{bmatrix} \tilde{\mathbf{Q}}_i & \mathbf{0} \\ \mathbf{0} & \mathbf{Z}_{x+w+v,i-1} \end{bmatrix}, \quad (43)$$

$$\mathbf{U}_i \triangleq \mathcal{A}_i^{l+1}\mathbf{\Theta}_l\mathcal{A}_i^{l+1^T} + \bar{\mathbf{B}}_{i,l}\mathbf{R}_i\bar{\mathbf{B}}_{i,l}^T, \quad (44)$$

$$\mathbf{F}_i \triangleq \mathcal{A}_i^{l+1}\mathbf{\Theta}_l\mathbf{C}_{i,l}^T + \bar{\mathbf{B}}_{i,l}\mathbf{R}_i\mathbf{H}_{i,l}^T, \quad (45)$$

$$\tilde{\mathbf{Q}}_i \triangleq \mathbf{D}_i\mathbf{Q}\mathbf{D}_i^T, \quad (46)$$

one accordingly gets $\mathbf{Z}_{x+w+v,i} = \mathbf{\Delta}_i + \mathbf{\Phi}_i$, where

$$\mathbf{\Phi}_i = \begin{bmatrix} \mathbf{C}_i\mathbf{U}_i\mathbf{C}_i^T & \mathbf{C}_i\mathbf{A}_i\mathbf{F}_{i-1} \\ \mathbf{F}_{i-1}^T\mathbf{A}_i^T\mathbf{C}_i^T & \mathbf{0} \end{bmatrix}. \quad (47)$$

With the matrix inversion lemma $(\mathbf{X}+\mathbf{Y})^{-1} = \mathbf{X}^{-1} - \mathbf{X}^{-1}(\mathbf{I}+\mathbf{Y}^{-1}\mathbf{X})^{-1}\mathbf{Y}^{-1}\mathbf{X}$ [22,23], the inverse of $\mathbf{Z}_{x+w+v,i}$ can be represented as

$$\mathbf{Z}_{x+w+v,i}^{-1} = \mathbf{\Delta}_i^{-1} - \mathbf{\Delta}_i^{-1}(\mathbf{I}+\mathbf{\Phi}_i\mathbf{\Delta}_i^{-1})^{-1}\mathbf{\Phi}_i\mathbf{\Delta}_i^{-1} . \quad (48)$$

Following this line, we decompose $\bar{\mathbf{B}}_{i,l}$ as $\bar{\mathbf{B}}_{i,l} = \left[\mathbf{B}_i \ \mathbf{A}_i\bar{\mathbf{B}}_{i-1,l}\right]$, the term $\bar{\mathbf{Z}}_{w,i}$ can be decomposed in an equivalent way as

$$\bar{\mathbf{Z}}_{w,i} = \left[\bar{\mathbf{B}}_{i,l}\mathbf{R}_i\bar{\mathbf{B}}_{i,l}^T\mathbf{C}_i^T \ \mathbf{A}_i\bar{\mathbf{Z}}_{w,i-1}\right] . \quad (49)$$

Using (49), we have

$$\begin{aligned} \mathcal{A}_i^{l+1}\mathbf{\Theta}_l\mathbf{C}_{i,l}^T &= \mathcal{A}_i^{l+1}\mathbf{\Theta}_l\left[(\mathbf{C}_i\mathcal{A}_i^{l+1})^T \ \mathbf{C}_{i-1,l}^T\right] \\ &= \left[\mathcal{A}_i^{l+1}\mathbf{\Theta}_l\mathcal{A}_i^{l+1^T}\mathbf{C}_i^T \ \mathcal{A}_i^{l+1}\mathbf{\Theta}_l\mathbf{C}_{i-1,l}^T\right] . \end{aligned} \quad (50)$$

Combining (49), (50) with (45), yields

$$\mathbf{F}_i = \left[\mathbf{U}_i\mathbf{C}_i^T \ \mathbf{A}_i\mathbf{F}_{i-1}\right] . \quad (51)$$

On the other hand, $\mathbf{U}_i$ can also be computed recursively as

$$\begin{aligned} \mathbf{U}_i &= \mathbf{A}_i\left(\mathcal{A}_{i-1}^{l+1}\mathbf{\Theta}_l\mathcal{A}_{i-1}^{l+1^T}\right)\mathbf{A}_i^T \\ &\quad + \left[\mathbf{B}_i\mathbf{R} \ \mathbf{A}_i\bar{\mathbf{B}}_{i-1,l}\mathbf{R}_{i-1}\right] \\ &\quad \times \left[\mathbf{B}_i^T \ \bar{\mathbf{B}}_{i-1,l}^T\mathbf{A}_i^T\right]^T \\ &= \mathbf{A}_i\mathbf{U}_{i-1}\mathbf{A}_i^T + \mathbf{B}_i\mathbf{R}\mathbf{B}_i^T . \end{aligned} \quad (52)$$

Using (48), (51) and (52), and taking into account the fact that $\hat{\mathbf{K}}_{i-1} = \mathbf{F}_{i-1}\mathbf{Z}_{x+w+v,i-1}^{-1}$, we have

$$\begin{aligned} \hat{\mathbf{K}}_i &= \left[\mathbf{U}_i\mathbf{C}_i^T \ \mathbf{A}_i\mathbf{F}_{i-1}\right] \\ &\quad \times \left[\mathbf{\Delta}_i^{-1} - \mathbf{\Delta}_i^{-1}(\mathbf{I}+\mathbf{\Phi}_i\mathbf{\Delta}_i^{-1})^{-1}\mathbf{\Phi}_i\mathbf{\Delta}_i^{-1}\right] \\ &= \left[\mathbf{U}_i\mathbf{C}_i^T\tilde{\mathbf{Q}}_i^{-1} \ \mathbf{A}_i\hat{\mathbf{K}}_{i-1}\right] \\ &\quad - \left[\mathbf{U}_i\mathbf{C}_i^T\tilde{\mathbf{Q}}_i^{-1} \ \mathbf{A}_i\hat{\mathbf{K}}_{i-1}\right] \\ &\quad \times \left(\mathbf{I}+\mathbf{\Phi}_i\mathbf{\Delta}_i^{-1}\right)^{-1}\mathbf{\Phi}_i\mathbf{\Delta}_i^{-1} . \end{aligned} \quad (53)$$

After some rearrangements, we arrive at

$$\hat{\mathbf{K}}_i = \left[\mathbf{U}_i\mathbf{C}_i^T\tilde{\mathbf{Q}}_i^{-1} \ \mathbf{A}_i\hat{\mathbf{K}}_{i-1}\right]\mathbf{S}_i^{-1}, \quad (54)$$

where

$$\mathbf{S}_i = \mathbf{I} + \mathbf{\Phi}_i\mathbf{\Delta}_l^{-1} = \begin{bmatrix} \mathbf{S}_{i11} & \mathbf{S}_{i12} \\ \mathbf{S}_{i21} & \mathbf{S}_{i22} \end{bmatrix}, \quad (55)$$

with

$$\mathbf{S}_{i11} = \mathbf{I} + \mathbf{C}_i\mathbf{U}_i\mathbf{C}_i^T\tilde{\mathbf{Q}}_i^{-1}, \quad (56)$$

$$\mathbf{S}_{i12} = \mathbf{C}_i\mathbf{A}_i\hat{\mathbf{K}}_{i-1}, \quad (57)$$

$$\mathbf{S}_{i21} = \mathbf{F}_{i-1}^T\mathbf{A}_i^T\mathbf{C}_i^T\tilde{\mathbf{Q}}_i^{-1}, \quad (58)$$

$$\mathbf{S}_{i22} = \mathbf{I} . \quad (59)$$

Using the Schur complement of $\mathbf{S}_{i11}$, the inverse matrix can be computed by

$$\mathbf{S}_i^{-1} = \begin{bmatrix} \bar{\mathbf{S}}_{i11}^{-1} & -\bar{\mathbf{S}}_{i11}^{-1}\mathbf{S}_{i12} \\ -\mathbf{S}_{i21}\bar{\mathbf{S}}_{i11}^{-1} & \mathbf{I}+\mathbf{S}_{i21}\bar{\mathbf{S}}_{i11}^{-1}\mathbf{S}_{i12} \end{bmatrix}, \quad (60)$$

where

$$\begin{aligned} \bar{\mathbf{S}}_{i11} &= \mathbf{S}_{i11} - \mathbf{S}_{i12}\mathbf{S}_{i22}^{-1}\mathbf{S}_{i21} \\ &= \mathbf{I} + \mathbf{C}_i\mathbf{N}_i\mathbf{C}_i^T\tilde{\mathbf{Q}}_i^{-1}, \end{aligned} \quad (61)$$

$$\mathbf{N}_i = \mathbf{U}_i - \mathbf{A}_i\hat{\mathbf{K}}_{i-1}\mathbf{F}_{i-1}^T\mathbf{A}_i^T . \quad (62)$$

Substituting (61) into (54), and using (52), (53) and (55), leads to

$$\hat{\mathbf{K}}_i = \left[\mathbf{G}_i \quad \mathbf{A}_i\hat{\mathbf{K}}_{i-1} - \mathbf{G}_i\mathbf{C}_i\mathbf{A}_i\hat{\mathbf{K}}_{i-1}\right], \quad (63)$$

where

$$\mathbf{G}_i = \mathbf{N}_i\mathbf{C}_i^T\left(\tilde{\mathbf{Q}}_i + \mathbf{C}_i\mathbf{N}_i\mathbf{C}_i^T\right)^{-1} . \quad (64)$$

According to (3), the OFIR estimate $\hat{\mathbf{x}}_i$ can be computed recursively by

$$\begin{aligned} \hat{\mathbf{x}}_i &= \left[\mathbf{G}_i \ \ \mathbf{A}_i\hat{\mathbf{K}}_{i-1} - \mathbf{G}_i\mathbf{C}_i\mathbf{A}_i\hat{\mathbf{K}}_{i-1}\right]\begin{bmatrix} \mathbf{y}_i \\ \mathbf{Y}_{i-1,l} \end{bmatrix} \\ &= \mathbf{A}_i\hat{\mathbf{x}}_{i-1} + \mathbf{G}_i\left(\mathbf{y}_i - \mathbf{C}_i\mathbf{A}_i\hat{\mathbf{x}}_{i-1}\right) . \end{aligned} \quad (65)$$

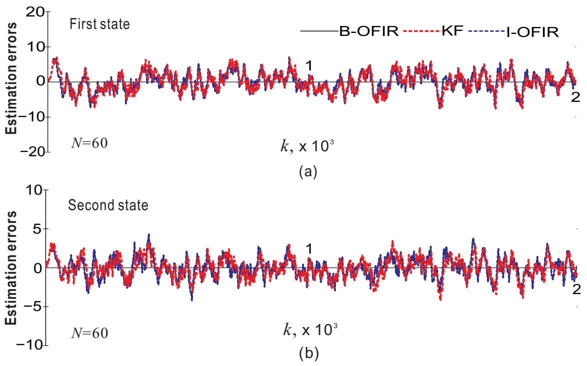

Figure 1: Estimation errors for an accurate model: (a) the first state and (b) the second state.

By setting $i = i+1$, we have

$$\mathbf{N}_{i+1} = \mathbf{U}_{i+1} - \mathbf{A}_i\hat{\mathbf{K}}_i\mathbf{F}_i^T\mathbf{A}_i^T\,. \tag{66}$$

Using the recursions of $\mathbf{U}_i$, $\hat{\mathbf{K}}_i$ and $\mathbf{F}_i$, arrive at

$$\mathbf{N}_{i+1} = \mathbf{A}_i\mathbf{N}_i\mathbf{A}_i^T + \mathbf{B}_i\mathbf{R}\mathbf{B}_i^T - \mathbf{A}_i\mathbf{G}_i\mathbf{C}_i\mathbf{N}_i\mathbf{A}_i^T\,, \tag{67}$$

where the property $\mathbf{U}_i = \mathbf{U}_i^T$ is employed. Substituting $\mathbf{G}_i$ with (64) and setting $i = i-1$, we find

$$\begin{aligned}\mathbf{N}_i &= \mathbf{A}_i\mathbf{N}_{i-1}\mathbf{A}_i^T + \mathbf{B}_i\mathbf{R}\mathbf{B}_i^T - \mathbf{A}_i\mathbf{N}_{i-1}\mathbf{C}_i^T \\ &\quad\times\left(\tilde{\mathbf{Q}}_i + \mathbf{C}_i\mathbf{N}_{i-1}\mathbf{C}_i^T\right)^{-1}\mathbf{C}_i\mathbf{N}_{i-1}\mathbf{A}_i^T\end{aligned} \tag{68}$$

which completes the proof.

5 Examples and Applications

In this section, we are going to show that the iterative OFIR form proposed can reduce the computation time considerably and without affecting the estimation accuracy.

Towards this end, we employ a two-state polynomial state space models (1) and (2) specified with $\mathbf{B}_k = [1, 1]^T$, $\mathbf{D}_k = 1$, $\mathbf{C}_k = [1, 0]$, and

$$\mathbf{A}_k = \begin{bmatrix} 1 & (1+d_k)\tau \\ 0 & 1 \end{bmatrix}, \tag{69}$$

where τ is a constant in unit of time, and d_k varies with time. Note that this kind of systems is commonly used to describe the "velocity jumps" in moving targets and "frequency jumps" in oscillators.

In the first simulation, we verify the fact that the iterative form described by theorem 1 can produce the same estimate as the batch form. The model parameter d_k is set as $d_k = 20$ if $160 \le k \le 200$ and $d_k = 0$ otherwise. The variances of the process noise and measurement noise are $\sigma_w^2 = 10^{-4}$ and $\sigma_v^2 = 10^2$,

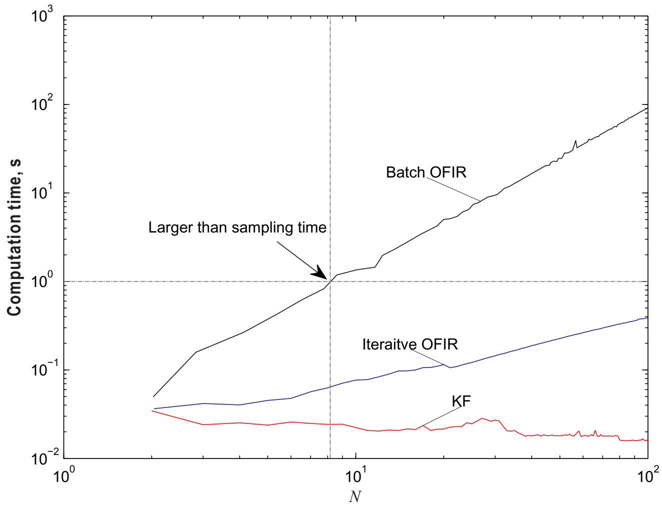

Figure 2: Computational times of different method as functions of estimation horizon N.

respectively. All the parameters of system model, including the variances of noises, are known completely in the entire estimation process. The estimation horizon was chosen as $N = 60$ and the batch OFIR (denoted as B-OFIR), iterative OFIR (I-OFIR), and KF estimates were obtained over 2000 subsequent points.

The estimation errors in a typical run are given in Fig. 1. As can be seen, both the I-OFIR and B-OFIR produce equal estimates and the KF and OFIR filter perform very close to each other. Of a special importance, we notice that the OFIR filter is low sensitive to the horizon length. Even with a time-invariant constant N, it can produce acceptable suboptimal estimates. In contrast, the performance of the UFIR filter strongly depends on the averaging horizon.

The computation time is another critical issue of FIR estimators which order is typically much higher than that of IIR estimators such as the KF. To show that the OFIR estimator can also operate fast, we next compare the batch and iterative OFIR filters to the KF in terms of the computation time using the same computer and software. The results are shown in Fig. 2. Definitely, the batch OFIR structure is worst, especially when N is set such that the computation time exceeds the sampling time; that is when $N \geqslant 8$ in Fig. 2. But the iterative OFIR algorithm demonstrates quite good properties for real-time applications, although it still loses to the KF. Note that the iterative OFIR algorithm can operate as fast as the KF if to organize iterations using parallel computing.

6 Conclusions

In this paper, we have developed an approach to provide fast iteration computation of OFIR estimates of time-variant systems using recursions. As a special feature of the algorithm proposed we notice that it is more general than the KF. Namely, the KF appears to be a special case of the iterative OFIR algorithm when the averaging horizon is infinite. Simulations have demonstrated that the iterative OFIR filter form operates much fasted than batch form without affecting the estimation accuracy.

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References:

[1] J. Salmi, A. Richter, and V. Koivunen, "Detection and tracking of MIMO propagation path parameters using state-space approach," *IEEE Trans. Signal Process.*, vol. 57, no. 4, pp. 1538–1550, Apr. 2009.

[2] I. Nevat and J. Yuan, "Joint channel tracking and decoding for BICM-OFDM systems using consistency test and adaptive detection selection," *IEEE Trans. Veh. Technol.*, vol. 58, no. 8, pp. 4316–4328, Oct. 2009.

[3] Y. S. Shmaliy, "Unbiased FIR filtering of discrete-time polynomial state-space models," *IEEE Trans. Signal Process.*, vol. 57, no. 4, pp. 1241–1249, Apr. 2009.

[4] P. S. Kim and M. E. Lee, "A new FIR filter for state estimation and its applications," *J. Comput. Sc. Techn.*, vol. 22, pp. 779–784, 2007.

[5] J. Fu, J. Sun, S. Lu, and Y. Zhang, "Maneuvering target tracking with modified unbiased FIR filter," *J. Beijing Univ. Aeronautics and Astronautics*, vol. 41, no. 1, pp. 72–82, 2015.

[6] Y. Levinson, L. Mirkin, L_2 optimization in discrete FIR estimation: Explotingg state-space structure, SIAM J. Control Optim. 51(1)(2013) 419–441.

[7] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*, New York: Academic, 1970.

[8] A. H. Jazwinski, "Limited memory optimal filtering," *IEEE Trans. Autom. Contr.*, vol. 13, no. 10, pp. 558–563, 1968.

[9] Y. S. Shmaliy, "Linear optimal FIR estimation of discrete time-invariant state-space models," *IEEE Trans. Signal Process.*, vol. 58, no. 6, pp. 3086–2010, Jun. 2010.

[10] C. K. Ahn and P. S. Kim, "Fixed-lag maximum likelihood FIR smoother for state-space models," *IEICE Electron. Express*, vol. 5, no. 1, pp. 11–16, 2008.

[11] C. K. Ahn, "Strictly passive FIR filtering for state-space models with external disturbance," *Int. J. Electron. Commun.*, vol. 66, no. 11, pp. 944–948, 2012.

[12] Y. S. Shmaliy and O. I. Manzano, "Time-variant linear optimal finite impulse response estimator for discrete state-space models," *Int. J. Adapt. Contrl Signal Process.*, vol. 26, no. 2, pp. 95–104, Sep. 2012.

[13] O. K. Kwon, W. H. Kwon, and K. S. Lee, "FIR filters and recursive forms for discrete-time state-space models," *Automatica*, vol. 25, no. 5, pp. 715–728, 1989.

[14] W. H. Kwon, P. S. Kim, and P. Park, "A receding horizon Kalman FIR filter for discrete time-invariant systems," *IEEE Trans. Autom. Contr.*, vol. 99, no. 9, pp. 1787–1791, 1999.

[15] Y. S. Shmaliy, "An iterative Kalman-like algorithm ignoring noise and initial conditions," *IEEE Trans. Signal Process.*, vol. 59, no. 6, pp. 2465–2473, Jun. 2011.

[16] P. S. Kim, "An alternative FIR filter for state estimation in discrete-time systems," *Digit. Signal Process.*, vol. 20, no. 3, pp. 935–943, May 2010.

[17] Y. S. Shmaliy, "Suboptimal FIR filtering of nonlinear models in additive white Gaussian noise," *IEEE Trans. Signal Process.*, vol. 60, no. 10, pp. 5519–5527, Oct. 2012.

[18] D. Simon and Y. S. Shmaliy, "Unified forms for Kalman and finite impulse response filtering and smoothing," *Automatica*, vol. 49, no. 6, pp. 1892–1899, Oct. 2013.

[19] F. R. Echeverria, A. Sarr, and Y. S. Shmaliy, " Optimal memory for discrete-time FIR filters in state-space," *IEEE Trans. Signal Process.*, vol. 62, pp. 557–561, Feb. 2014.

[20] S. Zhao, Y. S. Shmaliy, B. Huang and F. Liu, "Minimum variance unbiased FIR filter for discrete time-variant models," *Automatica*, 2015, 53: 355–361.

[21] S. Zhao, Y. S. Shmaliy, and F. Liu, "Fast computation of discrete optimal FIR estimates in white Gaussian noise," *IEEE Signal Process. Lett.*, vol. 22, no. 6, pp. 718–722, 2015.

[22] N. Dunford and J. T. Schwartz, *Linear Operators, Part I: General Theory*, John Wiley & Sons: New York, 1988.

[23] F. M. Callier and C. A. Desoer, *Linear System Theory*, Springer-Verlag: New-York, 1991.

[24] C. E. Souza, M. R. Gevers, and G. C. Goodwin, "Riccati equations in optimal filtering of nonstabilizable systems having singular state transition matrices," *IEEE Trans. Autom. Contr.*, vol. AC-20, no. 9, pp. 831–838, 1986.

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