Bounds for the electrical resistance for non-homogeneous two-dimensional conducting body

ISTVÁN ECSEDI, ATTILA BAKSA
Institute of Applied Mechanics
University of Miskolc
H-3515 Miskolc-Egyetemváros, Miskolc
HUNGARY

Abstract: - A mathematical model is developed to determine the steady-state current flow through in non-homogeneous isotropic conductor whose shape is a two-dimensional hollow domain. The corresponding electric boundary value problem is formulated using Maxwell's theory of electricity. The determination of the two-dimensional motion of charges is based on the concept of the electrical conductance. The Cauchy-Schwarz inequality is used to get the lower and upper bounds for the electrical conductance. The derived upper and lower bound formulae are illustrated by numerical examples.

Key-Words: - electrical conductance, electrical resistance, lower and upper bounds, steady-state

Received: November 12, 2021. Revised: October 16, 2022. Accepted: November 18, 2022. Published: December 15, 2022.

1 Introduction

Electrical resistance of an electrical conductor is a measure of the difficulty to pass a steady current through the conductor body. In this paper[1], the concept of electrical resistance and its inverse of electrical conductance are discussed. Some bounding formulae will be proven for the electric conductance. The well-known elementary form of Ohm's law states that when the conductor carries current I from a point P_1 at potential U_1 to point P_2 at potential U_2 then $U_2 - U_1 = RI$, where R is the resistance of the conductor between points P_1 and P_2. It depends only on the shape and temperature and the material of the conductor. The inverse of electric resistance is the electric conductance $G = 1/R$. This paper deals with the electric resistance of a two-dimensional isotropic non-homogeneous conductor body.

Examination of non-homogeneous structural elements is a very important task. The non-homogeneous isotropic hollow two-dimensional conductor is bounded by two closed curves ∂A_1 and ∂A_2 which have no common point. The current flow inside the conductor from the inner boundary curve ∂A_1 to the outer boundary curve ∂A_2. The potential on the inner boundary curve ∂A_1 is U_1 and the potential on the outer boundary curve ∂A_2 is U_2, $U_1 > U_2$. Two-side estimation will be proven for the electrical conductance of non-homogeneous isotropic two-dimensional conductor. The mathematical formalism follows the methods which are used in papers [1], [2], [3], [4]. In the Study of [1] upper and lower bounds are proven for electrical resistance of homogeneous isotropic ring-like axisymmetric conductor. The Study of [2] deals with the capacitance of two-dimensional cylindrical capacitor which consists of non-homogeneous dielectric materials.

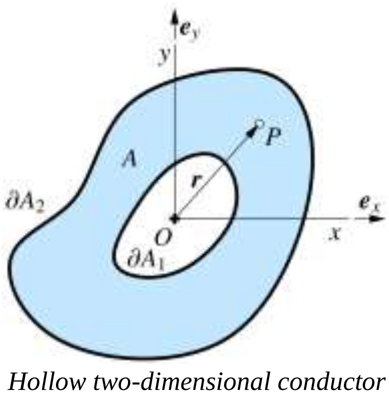

Fig. 1 *Hollow two-dimensional conductor body*.

Examples illustrate the applications of the derived bounding formulae of capacitance, [2]. A mathematical model for the steady-state heat transfer problem is developed in paper, [3]. Considered body of rotation is homogeneous and isotropic, [3]. In the study [4], by the application of Cauchy-Schwarz inequality upper and lower bounds are derived for the electrical resistance of a three-dimensional hollow conductor body.

Let us consider the steady-state motion of charges in a non-homogeneous hollow two-dimensional conductor body shown in Figure 1. The conductor body occupies the plane domain A and its boundary curves ∂A_1 and ∂A_2. The electric potential on the boundary curves ∂A_1 and ∂A_2 are prescribed, so that the following boundary conditions are valid, [5], [6], [7], [8].

$$U(\boldsymbol{r}) = U_1 \quad \boldsymbol{r} \in \partial A_1, \qquad U(\boldsymbol{r}) = U_2 \quad \boldsymbol{r} \in \partial A_2. \tag{1}$$

In equation (1) $\boldsymbol{r}$ is the position vector of an arbitrary point $P \in A \cup \partial A$. $\boldsymbol{r} = x\boldsymbol{e}_x + y\boldsymbol{e}_y$ (see Figure 1), x, y are Cartesian coordinates and $\boldsymbol{e}_x$ and $\boldsymbol{e}_y$ are the unit vectors of the coordinate system Oxy. According to Maxwell's theory [5], [6], [7], [8] the steady motion of the charges is described by the next equations

$$\boldsymbol{j} = \sigma \boldsymbol{E}, \quad \nabla \cdot \boldsymbol{j} = 0, \quad \boldsymbol{E} = -\nabla U. \tag{2}$$

Differential form of Ohm's law formulates that at constant temperature in isotropic conductor the current density vector $\boldsymbol{j}$ is proportional to the electric field vector $\boldsymbol{E}$. Here, $\sigma = \sigma(x, y) = \sigma(\boldsymbol{r})$ is the electric conductivity of the non-homogeneous hollow conductor body. In equation (2) ∇ is the del operator its representation on Cartesian coordinate system Oxy is

$$\nabla = \frac{\partial}{\partial x}\boldsymbol{e}_x + \frac{\partial}{\partial y}\boldsymbol{e}_y \tag{3}$$

and the dot between two vectors denotes the scalar product, [9]. From equation (2) it follows that

$$\sigma(\boldsymbol{r}) \vartriangle U + \nabla\sigma \cdot \nabla U = 0 \qquad \boldsymbol{r} \in A, \tag{4}$$

$$\vartriangle = \nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}. \tag{5}$$

The function $u = u(\boldsymbol{r})$ is defined by equation (6)

$$U(\boldsymbol{r}) = (U_1 - U_2)u(\boldsymbol{r}) + U_2, \qquad U_1 \neq U_2. \tag{6}$$

It is evident that $u = u(\boldsymbol{r})$ satisfies the following boundary value problem

$$\sigma(\boldsymbol{r}) \vartriangle u + \nabla\sigma \cdot \nabla u = 0 \qquad \boldsymbol{r} \in A, \tag{7}$$

$$u(\boldsymbol{r}) = 1 \quad \boldsymbol{r} \in \partial A_1, \quad u(\boldsymbol{r}) = 0 \quad \boldsymbol{r} \in \partial A_2. \tag{8}$$

The function $u = u(\boldsymbol{r})$ plays a crucial role in the expressions of electrical resistance and electrical conductance. An electric current in the two-dimensional conductor is the continuous passage of the current along the conductor. The constant potential difference between the closed curves ∂A_1 and ∂A_2 maintains the steady flow of electric current. The amount of charge following through curve ∂A_1 per unit time is I. The determination of I is based on the following equation

$$I = \int_{\partial A} \boldsymbol{j} \cdot \boldsymbol{n} \, \mathrm{d}s = (U_1 - U_2) \int_{\partial A_1} \sigma(\boldsymbol{r}) \boldsymbol{n} \cdot \nabla u \, \mathrm{d}s = (U_1 - U_2) \int_{\partial A_1} \sigma(\boldsymbol{r}) \frac{\partial u}{\partial \boldsymbol{n}} \, \mathrm{d}s. \tag{9}$$

In equation (9) the unit of I is $[\mathrm{A/m}]$ and $\boldsymbol{n}$ is the outer unit normal vector of the inner boundary curve ∂A_1 and ds is the arc element on ∂A_1. The electrical resistance R and the conductance $G = 1/R$ of the hollow two-dimensional conductor is defined, [6], [10]

$$R = \frac{U_1 - U_2}{I} = \frac{1}{\int_{\partial A_1} \sigma(\boldsymbol{r}) \frac{\partial u}{\partial \boldsymbol{n}} \mathrm{d}s} \tag{10}$$

$$G = \frac{I}{U_1 - U_2} = \int_{\partial A_1} \sigma(\boldsymbol{r}) \frac{\partial u}{\partial \boldsymbol{n}} \, \mathrm{d}s. \tag{11}$$

It is evident that

$$\nabla \cdot (\sigma(\boldsymbol{r}) \nabla u) = \sigma(\boldsymbol{r}) \vartriangle u + \nabla\sigma \cdot \nabla u = 0. \tag{12}$$

From equation (5) it follows that

$$\int_A u \nabla \cdot (\sigma(\boldsymbol{r}) \nabla u) \mathrm{d}A = \int_{\partial A} u \, \sigma(\boldsymbol{r}) \boldsymbol{n} \cdot \nabla u \, \mathrm{d}s - \int_A \sigma(\boldsymbol{r}) |\nabla u|^2 \mathrm{d}A = 0, \tag{13}$$

that is

$$\int_{\partial A_1} \sigma(\boldsymbol{r}) \frac{\partial u}{\partial \boldsymbol{n}} \, \mathrm{d}s = \int_A \sigma(\boldsymbol{r}) |\nabla u|^2 \, \mathrm{d}A. \tag{14}$$

Combination of equations (10) and (11) with equation (14) gives

$$R = \frac{1}{\int_A \sigma(\boldsymbol{r}) |\nabla u|^2 \, \mathrm{d}A}, \; G = \int_A \sigma(\boldsymbol{r}) |\nabla u|^2 \, \mathrm{d}A. \tag{15}$$

It should be mentioned that the units of R is $[\mathrm{V/Am}] = [\Omega/\mathrm{m}]$ and if

$$\nabla\sigma \cdot \nabla u = 0 \qquad \boldsymbol{r} \in A \tag{16}$$

then $u(\mathbf{r}) = u_0(\mathbf{r})$ where $u_0(\mathbf{r})$ is a unique solution of the following Dirichlet type boundary value problem

$$\vartriangle u_0 = 0 \quad \boldsymbol{r} \in A, \quad u_0(\boldsymbol{r}) = 1 \quad \boldsymbol{r} \in \partial A_1 \quad u_0(\boldsymbol{r}) = 0 \quad \boldsymbol{r} \in \partial A_2. \tag{17}$$

In this case

$$G = \int_A \sigma(\boldsymbol{r}) |\nabla u_0|^2 \, \mathrm{d}A. \tag{18}$$

There are several approximation methods to get the solution of the boundary value problem formulated in equations (8) and (9), most of which use the results of variational calculus, for example as Ritz method, finite element method, [10], [11], [12], [13], [14], [15]. Other numerical methods are also known and used, for example finite difference methods, method of weighted residuals, boundary element method, [16]. It is not the aim of this paper to give a detailed list of different analytical and numerical methods, which are used widespread in electric engineering calculation.

2 Upper bound for *G* and lower bound for *R*

Theorem 1 If the function of $F = F(\boldsymbol{r})$ which is continuously differentiable in $A \cup \partial A$ satisfies the boundary conditions

$$F(\boldsymbol{r}) = 1 \quad \boldsymbol{r} \in \partial A_1, \quad F(\boldsymbol{r}) = 0 \quad \boldsymbol{r} \in \partial A_2 \tag{19}$$

then the inequality relation for G

$$G \leq G_U = \int_A \sigma(\boldsymbol{r}) |\nabla F|^2 \, \mathrm{d}A \tag{20}$$

is valid and the sign of equality in bounding formula (20) is valid only if $F(\boldsymbol{r}) \equiv u(\boldsymbol{r})$.

Proof. The proof of inequality (20) is based on the Cauchy-Schwarz inequality

$$\left(\int_A \sigma(\boldsymbol{r}) \nabla F \cdot \nabla u \, \mathrm{d}A\right)^2 \leq \int_A \sigma(\boldsymbol{r}) |\nabla F|^2 \, \mathrm{d}A \int_A \sigma(\mathrm{r}) |\nabla u|^2 \, \mathrm{d}A. \tag{21}$$

A simple computation leads to the result

$$\int_A \sigma(\boldsymbol{r}) \nabla F \cdot \nabla u \, \mathrm{d}A = \int_A \nabla \cdot (F \sigma(\boldsymbol{r}) \nabla u) \mathrm{d}A - \int_A F \, \nabla \cdot (\sigma(\boldsymbol{r}) \nabla u) \mathrm{d}A = \int_{\partial A_1} \sigma(\boldsymbol{r}) \frac{\partial u}{\partial \boldsymbol{n}} \mathrm{d}s. \tag{22}$$

The combination of the inequality relation (21) with equation (22) and using the formulae of G gives the upper bound formula (20) for the electrical conductance. From the boundary condition (19) and that the equality in (21) is valid only if $F(\boldsymbol{r}) = \lambda u(\boldsymbol{r})$ where λ is an arbitrary constant.

3 Lower bound for *G* and upper bound for *R*

Theorem 2 Let $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{r})$ be a two-dimensional vector field defined in the hollow domain $A \cup \partial A$ which satisfies the following equation

$$\nabla \cdot (\sigma(\boldsymbol{r}) \boldsymbol{q}) = 0 \quad \boldsymbol{r} \in A, \tag{23}$$

in this case

$$G \geq G_L = \frac{\left(\int_{\partial A_1} \sigma(\boldsymbol{r}) \boldsymbol{n} \cdot \boldsymbol{q} \, \mathrm{d}s\right)^2}{\int_A \sigma(\boldsymbol{r}) \boldsymbol{q}^2 \, \mathrm{d}A}, \quad \int_A \boldsymbol{q}^2 \, \mathrm{d}A \neq 0. \tag{24}$$

In lower bound formula (24) equality is reached only if $\boldsymbol{q} = \lambda \nabla u$, where λ is an arbitrary constant which is differentiating from zero.

Proof. The proof of lower bound formula (24) is based on the following Cauchy-Schwarz inequality relation

$$\left(\int_A \sigma(\boldsymbol{r}) \nabla u \cdot \boldsymbol{q} \, \mathrm{d}A\right)^2 \leq \int_A \sigma(\boldsymbol{r}) |\nabla u|^2 \, \mathrm{d}A \int_A \sigma(\boldsymbol{r}) \boldsymbol{q}^2 \, \mathrm{d}A. \tag{25}$$

A simple computation gives

$$\int_A \sigma(\boldsymbol{r}) \nabla u \cdot \boldsymbol{q} \, \mathrm{d}A = \int_A \nabla \cdot (u \sigma(\boldsymbol{r}) \boldsymbol{q}) \, \mathrm{d}A - \int_A u \, \nabla \cdot (\sigma(\boldsymbol{r}) \boldsymbol{q}) \, \mathrm{d}A = \int_{\partial A_1} \sigma(\boldsymbol{r}) \boldsymbol{n} \cdot \boldsymbol{q} \, \mathrm{d}s. \tag{26}$$

Substitution of equation (26) into inequality (25) gives

$$\left(\int_{\partial A_1} \sigma(\boldsymbol{r}) \boldsymbol{n} \cdot \boldsymbol{q} \, \mathrm{d}s\right)^2 \leq G \int_A \sigma(\boldsymbol{r}) \boldsymbol{q}^2 \, \mathrm{d}A \tag{27}$$

that is

$$G \geq \frac{\left(\int_{\partial A_1} \sigma(\boldsymbol{r}) \boldsymbol{n} \cdot \boldsymbol{q} \, \mathrm{d}s\right)^2}{\int_A \sigma(r) q^2 \, \mathrm{d}A.} \tag{28}$$

Theorem 3 Let $f = f(\boldsymbol{r})$ be a non-identically constant function in $A \cup \partial A$, which satisfies the Laplace equation in A

$$\nabla \cdot \nabla f = \Delta f = 0 \qquad \boldsymbol{r} \in A. \tag{29}$$

The following lower bound formula is valid for G

$$G \geq G_L = \frac{\left(\int_{\partial A_1} \frac{\partial f}{\partial \boldsymbol{n}} \mathrm{d}s\right)^2}{\int_A \frac{|\nabla f|^2}{\sigma(\mathrm{r})} \mathrm{d}A.} \tag{30}$$

The proof of lower bound (29) can be obtained from inequality (28) with the under mentioned $\boldsymbol{q} = \boldsymbol{q}(\boldsymbol{r})$

$$\boldsymbol{q}(\boldsymbol{r}) = \frac{\nabla f}{\sigma(\mathbf{r})} \qquad \boldsymbol{r} \in A \cup \partial A. \tag{31}$$

4 Numerical Example

In the numerical examples the $Or\varphi$ plane polar coordinate system is used. The definition of the polar coordinates r and φ in terms of Cartesian coordinate is as follows

$$x = r\cos\varphi, \qquad y = r\sin\varphi, \tag{32}$$

$$r = \sqrt{x^2 + y^2}, \qquad \varphi = \arctan\frac{y}{x}. \tag{33}$$

4.1. Example 1

The boundary curves of the two-dimensional hollow conductor are an ellipse and a circle whose equations are (see Figure 2)

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1=0 \qquad (x,y)\in\partial A_2, \tag{34}$$

$$x^2+y^2-c^2=0 \qquad (x,y)\in\partial A_1. \tag{35}$$

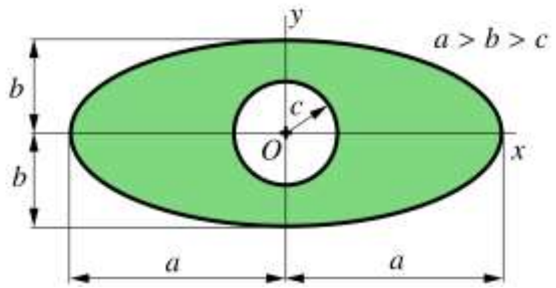

Fig. 2 *Conductor body bounded by a circle and an ellipse.*

The equation of the boundary curves in polar coordinates r, φ can be represented as

$$R_2(\varphi)=\frac{ab}{\sqrt{a^2\sin^2\varphi+b^2\cos^2\varphi}} \qquad 0\le\varphi\le 2\pi \tag{36}$$

$$R_1(\varphi)=c=\text{constant} \qquad 0\le\varphi\le 2\pi. \tag{37}$$

At first a homogeneous conductor is considered. The following numerical data are used

$$a=0.075\,\text{m},\quad b=0.065\,\text{m},\quad c=0.035\,\text{m}$$

$$\sigma=\sigma_0=7.69\times10^6\,\frac{1}{\Omega\text{m}}. \tag{38}$$

The bounding formulae (20) and (29) will be used. Let $F=F(r,\varphi)$ be

$$F(r,\varphi)=\frac{\ln\frac{R_2(\varphi)}{r}}{\ln\frac{R_2(\varphi)}{c}}. \tag{39}$$

in (20). Using function defined by equation (38) gives

$$G_U=7.084\,521\,374\times10^7\,\frac{1}{\Omega\text{m}}. \tag{40}$$

The substitution of the function

$$f(r,\varphi)=\ln r. \tag{41}$$

into the lower bound expression (29) the following result can be derived

$$G_L=7.022\,593\,443\times10^7\,\frac{1}{\Omega\text{m}}. \tag{42}$$

4.2 Example 2

In this example all data are the same as in Example 1 expecting the specific conductance σ which depends on the radial coordinate r

$$\sigma(\boldsymbol{r})=\sigma_0\frac{r}{g}, \qquad g=1.5\,\text{m}. \tag{43}$$

The same functions $F=F(\boldsymbol{r})$ and $f=f(\boldsymbol{r})$ are used to obtain upper and lower bounds for the non-homogeneous conductor body as in Example 1. The following results can be derived

$$G_U=2.375\,784\,710\times10^6\,\frac{1}{\Omega\text{m}}, \tag{44}$$

$$G_L=2.269\,373\,966\times10^6\,\frac{1}{\Omega\text{m}}. \tag{45}$$

4.3 Example 3

The boundary curves of the non-homogeneous hollow conductor body are two circles whose center points are different points as shown in Figure 3. The following numerical data are used

$$a=0.3\,\text{m}, b=0.4\,\text{m}, c=0.03\,\text{m}, R_1(\varphi)=a,$$

$$R_2(\varphi)=c\sin\varphi+\sqrt{b^2-c^2\cos^2\varphi} \quad 0\le\varphi\le2\pi$$

$$\sigma_0=7.69\times10^6\,\frac{1}{\Omega m}, \quad \sigma(r,\alpha)=\sigma_0\exp(\alpha r) \tag{46}$$

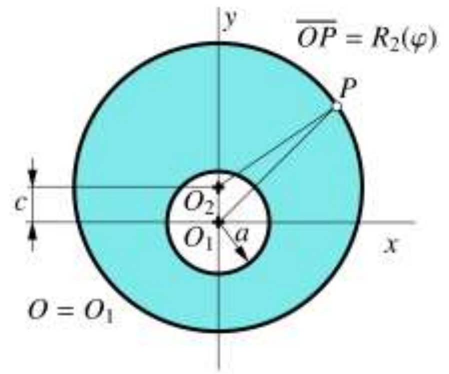

Fig. 3 *The hollow plane domain bounded by the two circles whose have different centers.*

In this example to obtain lower and upper bounds it is assumed that

$$F(r,\varphi)=\frac{\ln\frac{R_2(\varphi)}{r}}{\ln\frac{R_2(\varphi)}{a}} \quad a<r\le R_2(\varphi) \quad 0\le\varphi\le2\pi, \tag{47}$$

$$f(r,\varphi)=\ln r \qquad a\le r\le R_2(\varphi) \quad 0\le\varphi\le2\pi. \tag{48}$$

The unit of parameter α is [1/m]. A detailed computation gives the upper and lower bounds as a function of parameter α, $G_U=G_U(\alpha)$ and $G_L=G_L(\alpha)$. The plots of function $G_U=G_U(\alpha)$ and $G_L=G_L(\alpha)$ are shown in Figure 4.

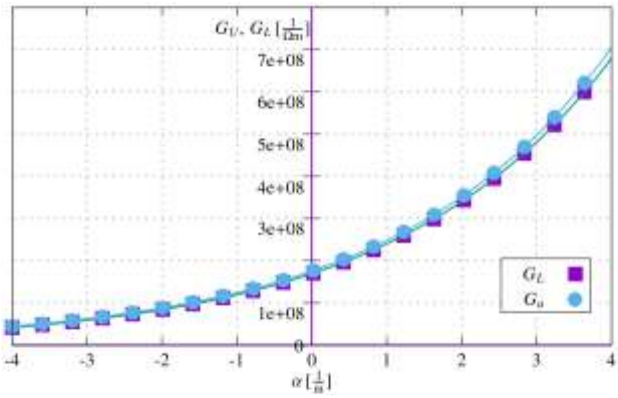

Fig. 4 *The graphs of functions* $G_U(\alpha)$ *and* $G_L(\alpha)$ *for* $-4 \le \alpha \le 4$.

The following numerical results can be obtained by the application bounding formulae for $\alpha_1 = 2\,[\frac{1}{\mathrm{m}}]$, $\alpha_2 = -2\,[\frac{1}{\mathrm{m}}]$

$$G_U(2) = 3.515\,266 \times 10^8\ \frac{1}{\Omega\mathrm{m}}. \qquad (49)$$

$$G_L(2) = 3.402\,427 \times 10^8\ \frac{1}{\Omega\mathrm{m}}. \qquad (50)$$

$$G_U(-2) = 8.827\,959 \times 10^7\ \frac{1}{\Omega\mathrm{m}}. \qquad (51)$$

$$G_L(-2) = 8.421\,377 \times 10^7\ \frac{1}{\Omega\mathrm{m}}. \qquad (52)$$

Figure 5 illustrates the graph of function $g(\alpha) = G_U(\alpha)/G_L(\alpha)$ for $-4 \le \alpha \le 4$.

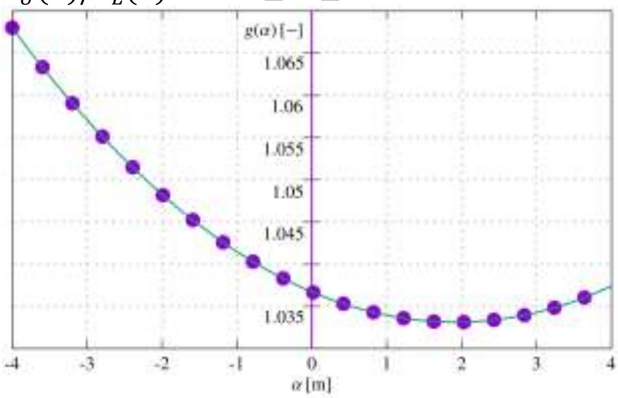

Fig. 5 *The plot of* $g(\alpha)$ *as a function of* α

for $-4 \le \alpha \le 4$.

4.4 Example 4

In this example a two-dimensional hollow conductor is considered whose inner and outer boundary curves are confocal ellipses (see Figure 6).

The common focus of boundary curves ∂A_1 and ∂A_2 is denoted by F_1 and F_2 and $\overline{OF_1} = \overline{OF_2} = c$. To develop the estimation formulae, it is necessary to introduce an orthogonal curvilinear coordinate system. The definition of the curvilinear coordinates ρ, ϑ is given by the following equations

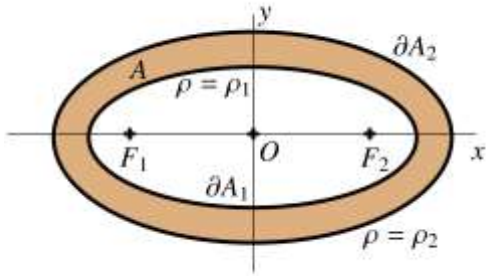

Fig. 6 *Conductor bounded by two confocal ellipses*.

$$x = \left(\rho + \frac{c^2}{4\rho}\right)\cos\vartheta \qquad y = \left(\rho - \frac{c^2}{4\rho}\right)\sin\vartheta$$

$$\rho_1 \le \rho \le \rho_2, \quad 0 \le \vartheta \le 2\pi. \qquad (53)$$

The semi axes of the boundary ellipses are

$$a_i = \rho_i + \frac{c^2}{4\rho_i}, \quad b_i = \rho_i - \frac{c^2}{4\rho_i} \quad (i = 1,2). \qquad (54)$$

Simple computations show that

$$a_1^2 - b_1^2 = a_2^2 - b_2^2 = c^2. \qquad (55)$$

It is assumed that the specific conductance σ depends on only the curvilinear coordinate ρ, that is $\sigma = \sigma(\rho)$. To obtain the upper bound for the conductance we use $F = F(\rho)$ in the formula (20) where

$$F(\rho_1) = 1, \quad F(\rho_2) = 0 \qquad 0 \le \vartheta \le 2\pi. \qquad (56)$$

The area element in curvilinear coordinate system $(\rho,\ \vartheta)$ is

$$\mathrm{d}A = H_\rho H_\vartheta \mathrm{d}\rho \mathrm{d}\vartheta, \qquad (57)$$

where

$$H_\rho^2 = \left(\frac{\partial x}{\partial \rho}\right)^2 + \left(\frac{\partial y}{\partial \rho}\right)^2, \qquad H_\vartheta^2 = \rho^2 H_\rho^2. \qquad (58)$$

The expression of the gradient of a function $Q = Q(\rho,\ \vartheta)$ in terms of curvilinear coordinates ρ and ϑ is

$$\nabla Q = \frac{1}{H_\rho}\frac{\partial Q}{\partial \rho}\boldsymbol{e}_\rho + \frac{1}{H_\vartheta}\frac{\partial Q}{\partial \vartheta}\boldsymbol{e}_\vartheta. \qquad (59)$$

where the unit vectors $\boldsymbol{e}_\rho$ and $\boldsymbol{e}_\vartheta$ are defined as

$$\boldsymbol{e}_\rho = \frac{1}{H_\rho}\left(\frac{\partial x}{\partial \rho}\boldsymbol{e}_x + \frac{\partial y}{\partial \rho}\boldsymbol{e}_y\right), \qquad (60)$$

$$\boldsymbol{e}_\vartheta = \frac{1}{H_\vartheta}\left(\frac{\partial x}{\partial \vartheta}\boldsymbol{e}_x + \frac{\partial y}{\partial \vartheta}\boldsymbol{e}_\vartheta\right). \qquad (61)$$

It is evident that

$$\sigma(\rho)|\nabla F|^2 \mathrm{d}A = \rho\sigma(\rho)\left(\frac{\mathrm{d}F}{\mathrm{d}\rho}\right)^2\ \mathrm{d}\rho\mathrm{d}\vartheta. \qquad (62)$$

Application of upper bound formula (20) gives

$$G \leq G_U = 2\pi \int_{\rho_1}^{\rho} \rho\sigma(\rho) \left(\frac{\mathrm{d}F}{\mathrm{d}\rho}\right)^2 \mathrm{d}\rho. \quad (63)$$

By the application of the known results of variational calculus, [17],[18] it can be pointed out that the upper bound (62) is the sharpest if

$$F(\rho) = 1 - \frac{\int_{\rho_1}^{\rho} \frac{\mathrm{d}\lambda}{\lambda\sigma(\lambda)}}{\int_{\rho_1}^{\rho_2} \frac{\mathrm{d}\rho}{\rho\sigma(\rho)}} \quad \rho_1 \leq \rho \leq \rho_2. \quad (64)$$

Substitution of equation (63) into upper bound formula (62) gives

$$G \leq G_U = \frac{2\pi}{\int_{\rho_1}^{\rho_2} \frac{\mathrm{d}\rho}{\rho\sigma(\rho)}}. \quad (65)$$

To get the lower bound for G a solution of the Laplace equation

$$\Delta f = \frac{1}{\rho H_\rho^2}\left[\frac{\partial}{\partial\rho}\left(\rho\frac{\partial f}{\partial\rho}\right) + \frac{\partial}{\partial\vartheta}\left(\frac{1}{\rho}\frac{\partial f}{\partial\vartheta}\right)\right] = 0. \quad (66)$$

must be computed. It is assumed that $f = f(\rho)$, that is, f does not depend on the curvilinear coordinate ϑ. It can be proven that

$$f(\rho) = \ln\rho \quad \rho_1 \leq \rho \leq \rho_2 \quad 0 \leq \vartheta \leq 2\pi \quad (67)$$

is a regular harmonic function in the double connected plane domain A. Substitution of function given by equation (66) in the lower bound formula (29) yields the result

$$G \geq G_L = \frac{2\pi}{\int_{\rho_1}^{\rho_2} \frac{\mathrm{d}\rho}{\rho\sigma(\rho)}}. \quad (68)$$

Comparison the expressions of upper and lower bound gives

$$G_U = G_L = G = \frac{2\pi}{\int_{\rho_1}^{\rho_2} \frac{\mathrm{d}\rho}{\rho\sigma(\rho)}}, \quad (69)$$

that is, the formula (68) is the exact value of the conductance.

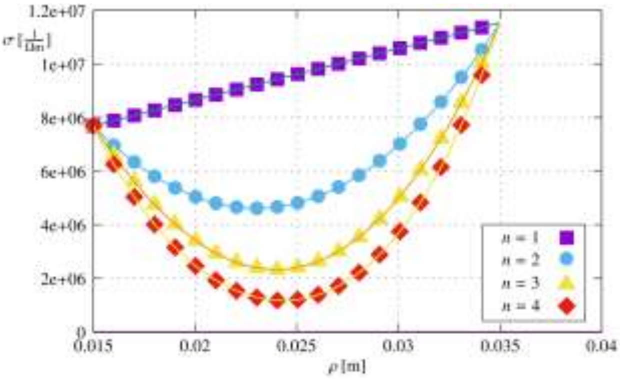

Fig. 7 *The plots of the specific conductance for* $n = 1,2,3,4$.

Let the material of the conductor body be functionally graded which specific conductance obeys the following equation

$$\sigma(\rho, n) = \sigma_1 \left(\frac{\rho_2 - \rho}{\rho_2 - \rho_1}\right)^n + \sigma_2 \left(\frac{\rho - \rho_1}{\rho_2 - \rho_1}\right)^n. \quad (70)$$

The plots of the function $\sigma(\rho, n)$ for some different values of material parameter n are shown in Figure 7.

The following data was used for computing the results in Figure 7.

$$\rho_1 = 0.015\,\mathrm{m}, \quad \rho_2 = 0.035\,\mathrm{m}$$

$$\sigma_1 = 7.69 \times 10^6 \frac{1}{\Omega m}, \sigma_2 = 1.1535 \times 10^7 \frac{1}{\Omega\mathrm{m}}. \quad (71)$$

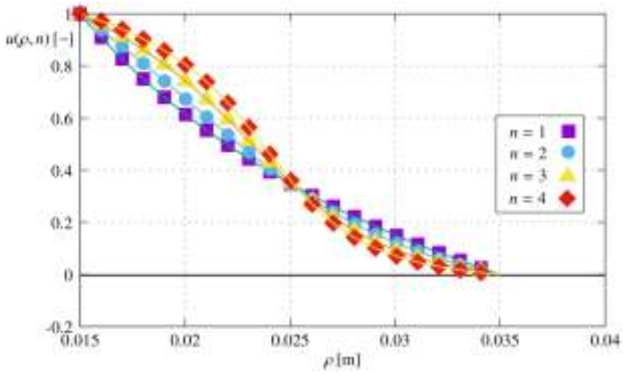

Fig. 8 *The graphs of function* $u(\rho, n)$ *as a function of* ρ *for* $n = 1,2,3,4$.

In Figure 8 the plots of the solution of the boundary value problem are presented for four different values of the material parameter.

The dependence of the conductance from the material parameter is shown in Figure 9 for $1 \leq n \leq 5$.

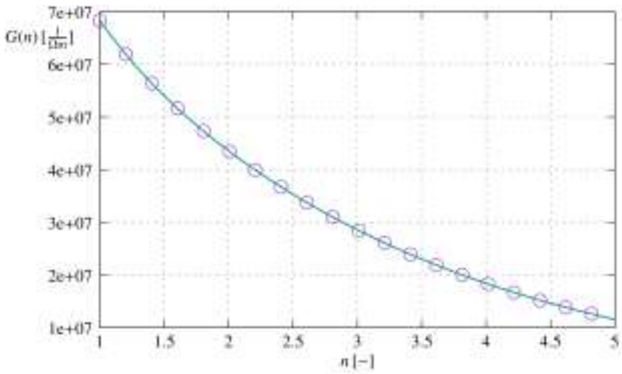

Fig. 9 *The plot of the function* $G = G(n)$ *for* $1 \leq n \leq 5$.

5 Conclusion

Upper and lower bounds are proven a two-dimensional hollow non-homogeneous isotropic conductor body by the application of Cauchy-Schwarz inequality. Several examples illustrate the applications of the derived bounding formulae.

The obtained numerical results can be used to check the numerical solutions obtained by finite element method, boundary element method, finite difference method and by other numerical methods.

References:

[1] I. Ecsedi, Á.J. Lengyel, A. Baksa, D. Gönczi, Bounds for the electrical resistance for homogeneous conducting body of rotation, *Multidiszciplináris tudományok*, Vol. 11, No. 5, 2021, pp. 104–121.

[2] I. Ecsedi, Á.J. Lengyel, Bounding capacitance of cylindrical capacitor with non-homogeneous dielectric material, *WSEAS Transactions on Electronics*, Vol. 12, 2021, pp. 125-131.

[3] I. Ecsedi, Á.J. Lengyel, D. Gönczi, Bounds for the thermal conductance of body of rotation, *Int. Rev. Model Simulations*, Vol. 13, No. 3, 2021, pp. 185-195.

[4] I. Ecsedi, A. Baksa, Bounds for the electrical resistance for non-homogeneous conducting body, *Pollac Periodica*, Vol. No. 2022, https://doi.org/10.1556/606.2022.00621

[5] J.D. Jackson, *Classical Electrodynamics*, 3^{rd} *ed.*, Wiley and Sons, New York, 1998.

[6] K. Simony, *Foundations of Electrical Engineering: Fields–Networks–Waves*, Pergamon Press, Oxford, 1963.

[7] L.D. Landau, E.M. Lifshitz, *Electrodynamics of Continuous Media*, Pergamon Press, Oxford, 1963.

[8] L. Solymar, *Lectrues on Electromagnetic Theory: A Short Course for Engineers*, Oxford University Press, Oxford, 1964.

[9] G.A. Korn, T.M. Korn, *Handbook for Scientists and Engineers*, D. von Nostrand, New York, 1961.

[10] P. Hammond, *Energy Method in Electromagnetism*, Clarendon Press, Oxford, 1997.

[11] P.P. Sisvester, R.L. Ferrari, *Finite Elements for Electrical Engineers*, Cambridge University Press, Cambridge 1983.

[12] A. Iványi, *Continuous and Discrete Simulations in Electrodynamics (in Hungarian)*, Akadémiai Kiadó, Budapest, 2003.

[13] A. Iványi, Variational methods in static electric field, (in Hungarian), *Elektrotechnika*, Vol. 71, 1978, pp. 21–25.

[14] A. Iványi, Determination of static and stationary electric fields by variational calculus, *Period. Polytech. Electr. Eng.*, Vol. 23, 1979, pp. 201–208.

[15] A. Iványi, R-functions in electromagnetism, *Technical Report*, No. TUPs-TR-93-EE08, Budapest, 1993.

[16] A.P. Boresi, K.P. Chang, S. Saigal, *Approximate Solution Methods in Engineering Mechanics*, New York, John Wiley and Sons, 2003.

[17] R. Weinstock, *Calculus of Variations*, McGraw-Hill, New York, 1952. I.M. Gelfand,

[18] S.V. Fomin, *Calculus of Variations*, Dover Publ., New York, 2000.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

István Ecsedi and Attila Baksa carried out the investigation and the formal analysis. István Ecsedi has implemented the algorithm for all the examples. Attila Baksa was responsible for the validation and for the visualization of the results. Both authors have been writing the paper with original draft, review and editing.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself
The author(s) received no financial support for the research, authorship, and/or publication of this article. Furthermore, on behalf of all authors, the corresponding author states that there is no conflict of interest.