A Special Note According to Possible Applications of Fractional-Order Calculus for Various Special Functions

HÜSEYİN IRMAK
Department of Computer Engineering,
Faculty of Engineering & Architecture,
İstanbul Nişantaşı University,
TR- 34481742, Sarıyer, Istanbul,
TURKEY

Abstract: - The main aim of this special study is to recall certain information about fractional (arbitrary) order calculus, which has wide and fruitful applications in science and engineering. Then, it aims to consider various essential definitions related to fractional order integrals and derivatives for stating and proving some results, as well as to present some of their possible applications to the attention of related researchers.

Key-Words: - Error functions, derivative(s) of fractional-order, differential equations of fractional order, series expansions, analytic functions, regions in the complex plane, special functions.

Received: March 7, 2024. Revised: September 8, 2024. Accepted: October 11, 2024. Published: November 6, 2024.

1 Introduction and Certain Special Information

In this first section, specific information about certain calculus of fractional order and related applications will be presented.

The concept of operators of fractional order was introduced almost simultaneously with the development of the classical ones.

In light of the essential information provided by written mathematical literature, as special information, the first known reference can be found in the correspondence between G. W. Leibniz and Marquis de l'Hospital in the year 1695, where the question of the meaning of the semi-derivative was raised. This special question consequently attracted the interest of many well-known mathematicians, including Euler, Liouville, Laplace, Riemann, Grünwald, Letnikov, and many others. Since the 19th century, the theory of fractional calculus has developed rapidly, mostly as a foundation for a number of applied disciplines, including fractional geometry, fractional differential equations, and fractional dynamics.

The extensive applications of fractional order calculus are very broad nowadays. It is safe to say that almost no discipline of modern engineering and science remains untouched by the tools and techniques of fractional calculus. For example, wide and fruitful applications can be found in rheology, viscoelasticity, acoustics, optics, chemical and statistical physics, robotics, control theory, electrical and mechanical engineering, bioengineering, and more.

In fact, one could argue that real-world processes are generally fractional order systems. The main reason for the success of its applications is that these new models of fractional order are often more accurate than integer-order ones, as they provide more degrees of freedom than the corresponding classical models. One of the intriguing beauties of the subject is that fractional derivatives (and integrals) are not local (or point) quantities.

All operators of fractional order type also consider the entire history of the process, thus being able to model the non-local and distributed effects often encountered in natural and technical phenomena. The calculus of fractional (arbitrary) order is therefore an excellent set of tools for describing the memory and hereditary properties of various materials and processes.

In addition to theoretical studies, in terms of the various application areas highlighted above, fractional order calculus is a frequently encountered research area for both functions of real independent variables and complex functions of independent variables. In various applications, the relevant fractional order calculus expression, especially fractional order derivatives, gains importance.

As various references, each of the references given in [1], [2], [3], [4] and [5] is a main source pertaining to fractional-order calculus, and the references given in [6], [7], [8], [9], [10], [11], [12], [13], [14], [15], [16], [17], [18] and [19] are sources for various applications of fractional-order calculus in different scientific fields. Especially since our investigation is related to the theory of complex functions, the studies given in [20], [21] and [22] are comprehensive references for those functions. In addition, for various special functions as well as transformation theories and their applications, the works in [23], [24], [25], [26], [27], [28], [29], [30], [31] and [32] can be provided as different types of references.

2 Definitions, Remarks, Properties and Special Examples

In this section of the research, we will cover various types of functions with complex variables. Now, let us move on to the following section to provide some relevant definitions.

Firstly, the familiar notations $\mathbb{N}$, $\mathbb{R}$ and $\mathbb{C}$ denote the set of natural numbers, the set of real numbers and the set of complex numbers, respectively.

We now begin by stating and introducing the fundamental definitions related to fractional-order integrals and fractional-order derivatives of functions with complex (*or* real) variables.

Definition 1. For a (complex) function $v \coloneqq v(z)$, the fractional integral of (arbitrary) order τ is denoted by

$$\mathfrak{D}_z^{-\tau}[v(z)] \equiv \mathfrak{D}_z^{-\tau}[v](z),$$

and also defined by:

$$\mathfrak{D}_z^{-\tau}[v(z)] = \frac{1}{\Gamma(\tau)} \int_0^z \frac{v(s)}{(z-s)^{1-\tau}} ds, \qquad (1)$$

where $\tau > 0$ and the function with complex variable v is analytic in any simply connected region of the z-plane including the origin, and the multiplicity of $(z-s)^{\tau-1}$ is removed by necessitating $log(z-s)$ to be a real number *when* $z-s>0$.

Definition 2. For a (complex) function $v := v(z)$, the fractional derivative of (arbitrary) order τ is denoted by

$$\mathfrak{D}_z^{\tau}[v(z)] \equiv \mathfrak{D}_z^{\tau}[v](z) \equiv \frac{d^\tau}{dz^\tau}[v]$$

and, also described as:

$$\mathfrak{D}_z^{\tau}[v(z)] = \begin{cases} \frac{1}{\Gamma(1-\tau)} \frac{d}{dz} \int_0^z \frac{v(z)}{(z-s)^{\tau}} ds & (0 \le \tau < 1) \\ \frac{d^\ell}{dz^\ell}\left\{\mathfrak{D}_z^{\tau-\ell}[v](z)\right\} & (0 \le \tau - \ell < 1) \end{cases} \qquad (2)$$

where $\ell \in \mathbb{N}$ and the analytic function $v(z)$ is constrained, and the multiplicity of $(z-s)^{-\tau}$ is extinct as in the first definition (just above). For these definitions above, one may refer to the essential works given in [1], [5], [10], [22] and [27].

We specifically note that the value of the parameter τ, *which* relates to the fractional order of the mentioned integrals and derivatives and their various possible applications, can also be any complex number. Due to the related hypotheses, it is necessary that the real part of the parameter τ be greater than zero.

In addition, *when* the parameter τ is any complex number, its real part must be greater than zero, with $j := 1 + [\![\Re e(\varphi)]\!]$, *where* the familiar notation $[\![\cdot]\!]$ denotes the greatest integer function in classical mathematics.

For additional information regarding the definitions described in Definitions 1 and 2, one may refer to some of the essential earlier results presented in [2], [4], [26], [27] and [32].

For our investigation, the following special information (*or* assertions), *which* are directly related to these main definitions and their possible applications, is required.

Remark 1. As a result of a straightforward examination of Definitions 1 and 2, the accuracy of the well-known properties related to scalar multiplication and linearity for these definitions is readily apparent. The details are omitted here.

We note that the value of the parameter τ, which represents the fractional-order calculus, can also be any complex number. Due to the hypotheses, it is necessary that the real part of τ be is greater than zero.

As some special information, we want to constitute the following assertions as some remarks by considering the power function with complex variable z given by

$$v(z) \coloneqq z^s \qquad (3)$$

for some $s \in \mathbb{R}$ with $s > -1$ (and, of course, $\Re e(s) > -1$ *when* selecting $s \in \mathbb{C}$), which are just below.

Remark 2. In view of the special information given by (1), (2) and (3), and also by making use of change of the variable $\tau = zt$, its fractional integral of real order $\tau\,(\tau>0)$ can be easily determined as the relations given by:

$$\mathfrak{D}_z^{-\tau}[z^s]=\frac{1}{\Gamma(\tau)}\int_0^z t^s\,(z-t)^{\tau-1}dt$$
$$=\int_0^1 \tau^s(1-\tau)^{\tau-1}d\tau$$
$$=\frac{\Gamma(s+1)}{\Gamma(s+\tau+1)}z^{s+\tau} \qquad (4)$$

for some value τ with $\tau>0$.

Remark 3. By the help of the relevant power function given by (3), its fractional derivative of order $\tau\ (0\le\tau<1)$ can be easy determined as the elementary result consisting of the relationships given by:

$$\mathfrak{D}_z^{\tau}[z^s]\equiv\frac{1}{\Gamma(1-\tau)}\frac{d}{dz}\left(\int_0^z \tau^s\,(z-t)^{-\tau}dt\right)$$
$$=\frac{1}{\Gamma(1-\tau)}\frac{d}{dz}[z^{s-\tau+1}$$
$$\times\int_0^1 \tau^s\,(1-\tau)^{-\tau}]d\tau$$
$$=\frac{\Gamma(s+1)}{\Gamma(s-\tau+1)}\,z^{s-\tau}, \qquad (5)$$

where $0\le\tau<1$.

Remark 4. In the light of both the definition given in (2) and the elementary result given by (5), its fractional derivatives of order $\ell+\tau$ can be also determined as the elementary form given by:

$$\mathfrak{D}_z^{\ell+\tau}[z^s]\equiv\frac{d^\ell}{dz^\ell}\{\mathfrak{D}_z^{\tau}[z^s]\}$$
$$=\frac{\Gamma(s+1)}{\Gamma(s-\tau+1)}\frac{d^\ell}{dz^\ell}\{z^{s-\tau}\}$$
$$=\frac{\Gamma(s+1)}{\Gamma(s-\ell-\tau+1)}z^{s-\ell-\tau}, \qquad (6)$$

where $0\le\tau-\ell<1$ and $\ell\in\mathbb{N}_0:=\mathbb{N}\cup\{0\}$.

By means of the extensive information between (1) and (6), the special assertions given by:

$$\mathfrak{D}_z^{0}[v(z)]=v(z)\,,$$
$$\lim_{\tau\to1^-}\mathfrak{D}_z^{\tau}[v(z)]=v'(z)$$

and

$$\lim_{\tau\to0^+}\mathfrak{D}_z^{\ell+\tau}[v(z)]=\mathfrak{D}_z^{\ell}[v(z)]$$
$$=\frac{d^\ell}{dx^\ell}\big(v(z)\big)$$

for all $\ell\in\mathbb{N}_0$, and also the elementary-special results given by:

$$15\sqrt{\pi z}\,\mathfrak{D}_z^{-1/2}[z^2]=16z^3,$$
$$15\sqrt{\pi z}\,\mathfrak{D}_z^{-3/2}[z]=8z^3,$$
$$3\sqrt{\pi z}\,\mathfrak{D}_z^{1/2}[z^2]=4z^2 \qquad (7)$$

and

$$\sqrt{\pi z}\,\mathfrak{D}_z^{3/2}[z^2]=4z \qquad (8)$$

can easily be constituted as some more special examples.

In particular, it is also possible to obtain several elementary results related to various types of fractional-order differential equations. In the simplest term, as a simple example, for an appropriate function $V:=V(z)$, the fractional-order equation being of

$$\sqrt{\pi}\,\mathfrak{D}_z^{3/2}[V]+3\sqrt{\pi}\,\mathfrak{D}_z^{1/2}[V]$$
$$=4(1+2z^2)\sqrt{z} \qquad (9)$$

can easily be designated by the help of a combining of the elementary results given by (7) and (8).

For these types of equations, as examples, one can refer to the essential works [1], [7], [18], [22], [28] and [32].

3 Final Remarks

As noted, in the previous two sections, some special information about fractional-order calculus was first presented, followed by a number of definitions, fundamental properties, and particular examples related to fractional-order calculus. In this final section, we will present special information consisting of various results and suggestions directly related to the main goal of our research, which involves complex-type special functions

For those results and possible implications, firstly, we want to center upon only two complex functions *which* are called as the complex error function and the complementary complex error function, respectively. These special functions also have important roles in nearly all sciences and technology.

For example, for the main (complex) error functions in the familiar forms:

$$erf(z) \quad\text{and}\quad erfc(z),$$

are defined by

$$erf(z)=\frac{2}{\sqrt{\pi}}\int_0^z e^{-q^2}dq \qquad (10)$$

and

$$erfc(z)=\frac{2}{\sqrt{\pi}}\int_z^\infty e^{-q^2}dq\,, \qquad (11)$$

where $z\in\mathbb{C}$.

These two fundamental functions both contain many general properties and have extensive relationships with various complex-type special functions. Especially, by the help of the equivalent assertions given by

$$\sqrt{\pi} = \int_{-\infty}^{\infty} e^{-q^2} dq$$
$$= 2\int_0^{\infty} e^{-q^2} dq$$
$$= \sqrt{\pi}\left(\frac{2}{\sqrt{\pi}}\int_0^z e^{-q^2} dq + \frac{2}{\sqrt{\pi}}\int_z^{\infty} e^{-q^2} dq\right)$$

the following main relationship can easily be achieved:

$$erf(z) + erfc(z) = 1 \quad (z \in \mathbb{C}), \qquad (12)$$

when one considers the definitions introduced in (10) and (11).

For additional properties and relationships, the earlier studies presented in [21], [22], [23], [24], [25] and [30] can also be examined.

Furthermore, for the (complex) error function presented in (10), one can consider the definition given in (2) and make use of the Taylor-Maclaurin series expansion, *which* is quite useful in approximation theory, provided by

$$e^{-q^2} = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!} q^{2j},$$

the series expansions of the special function $erf(z)$ with

$$erf(z) = \sum_{k=0}^{\infty} \omega(k)\, z^{2k+1} \qquad (13)$$

can easily be obtained for some $z \in \mathbb{C}$ and for all $k \in \mathbb{N}_0$:

$$\omega(k) := \frac{2}{\sqrt{\pi}} \frac{(-1)^k}{(2k+1)k!}\,. \qquad (14)$$

Additionally, using the information from the assertions given in (5) and (13), one can easily arrive at:

$$\mathfrak{D}_z^{\tau}[erf(z)]$$
$$= \mathfrak{D}_z^{\tau}\{\sum_{k=0}^{\infty} \omega(k) z^{2k+1}\}$$
$$= \sum_{k=0}^{\infty}\left[\omega(k)\, \mathfrak{D}_z^{\tau}\left(z^{2k+1}\right)\right] \qquad (15)$$
$$= z^{-\tau}\sum_{k=0}^{\infty}\left(\frac{(2k+1)!}{\Gamma(2k-\tau+2)}\omega(k)(-z)^{2k+1}\right),$$

where $z \in \mathbb{C}$ and $\omega(k)$ denotes the function defined by (14).

At the same time, along with its undeniable importance in approximation theory, and in light of the information presented in (12) and (15), it is easy to obtain the series expansion of the function $erfc(z)$ and also its fractional-order derivative(s), *which* are:

$$erfc(z) = 1 - \sum_{k=0}^{\infty} \omega(k)\, z^{2k+1}$$

and also

$$\mathfrak{D}_z^{\tau}[erfc(z)]$$
$$= \mathfrak{D}_z^{\tau}[1 - erf(z)]$$
$$= \left(\frac{1}{\Gamma(1-\tau)} - \sum_{k=0}^{\infty} \Omega(k)\, z^{2k+1}\right) z^{-\tau}$$

for some $z \in \mathbb{C} - \{0\}$ and for all $k \in \mathbb{N}_0$:

$$\Omega(k) := \frac{(2k+1)!}{\Gamma(2k-\tau+2)}\omega(k)\,,$$

where $\omega(k)$ is the mentioned function defined by (14).

Secondly, inspired by the special fractional-order differential equation given in (9), we can both increase the number of such equations and determine their solutions.

Additionally, in light of information related to operator theory, various applications of integral operators and their inverses can be considered for all possible results. For this extensive investigation, we will highlight only one example and present other potential investigations for the attention of related researchers. For a simple example, consider the possible solution of the initial value problem of fractional order in the special form:

$$\begin{cases} \mathfrak{D}_z^{1}[V] + \mathfrak{D}_z^{1/2}[V] + 2V = 0 \\ V(0) = 1 \end{cases},$$

where $V := V(z)$, one can first obtain the explicit form of the series expansion of

$$V(z) = \sum_{v=0}^{\infty} z^{v/2}\omega_v\,,$$

and then focus on the indicated research.

In conclusion, the mathematical literature contains a wealth of information and applications related to other special functions and their implications. For additional information, we recommend that our readers review the earlier studies referenced in [4], [9], [21], [23], [24], [30], [32] and [33].

References:

[1] K. S. Miller and B. Ross, *"An introduction to the fractional calculus and fractional differential equations,"* Wiley, New York, 1993.

[2] K. B. Oldham and J. Spanier, *"The fractional calculus: theory and applications of differentiation and integration to arbitrary order,"* Academic Press, San Diego, 1974.

[3] S. Das, *Functional fractional calculus.* Springer, Berlin, 2011.

[4] D. Valério, J. T. Machado and V. Kiryakova, "Some pioneers of the applications of fractional calculus," *Fractional Calculus and Applied Analysis,* Vol. 7, No. 2, 2014, pp. 552-578, DOI: 10.2478/s13540-014-0185-1.

[5] H. M. Srivastava and S. Owa, *"Univalent Functions, Fractional Calculus and Their Applications,"* Halsted Press, John Wiley and Sons, New York, Chieschester, Brisbane, Toronto, 1989.

[6] I. Podlubny, "Fractional-order systems and PID-controllers," *IEEE Transactions on Automatic Control*, Vol. 44, No. 1, 1999, pp. 208-214.

[7] K. Diethelm, *"The analysis of fractional differential equations: An application-oriented exposition using differential operators of Caputo type,"* Springer, New York, 2010.

[8] R. Hilfer, "*Applications of fractional calculus in physics,*" World Scientific, Singapore, 2000.

[9] R. L. Magin, *"Fractional calculus in bioengineering,"* Begell House Publishers, Redding, 2006.

[10] H. Irmak, "Some operators of fractional calculus and their applications regarding various complex functions analytic in certain domains," *Studia Universitatis Babeş-Bolyai Mathematica,* Vol. 69, No. 3, 2024, pp. 471-482.

[11] C. P. Li and F. H. Zeng, "Numerical methods for fractional calculus," CRC Press, Boca Raton, 2015.

[12] I. Podlubny, "Geometric and physical interpretations of fractional integration and differentiation," *Fractional Calculus and Applied Analysis,* Vol. 5, No. 4, 2001, pp. 230-237.

[13] A. Tapadar, F. A. Khanday, S. Sen and A. Adhikary, "Fractional calculus in electronic circuits: A review," *Fractional Order System,* Vol. 1, 2022, pp. 441-482.

[14] C. Lubich, "Discretized fractional calculus," *SIAM Journal on Mathematical Analysis,* Vol. 17, No. 3, 1986, pp. 704-719.

[15] K. Diethelm and N. J. Ford, "Multi-order fractional differential equations and their numerical solution," *Applied Mathematics and Computation,* Vol. 154, No. 3, 2004, pp. 621-640.

[16] Z. Li, L. Liu, S. Dehghan, Y. Chen and D. Xue, "A review and evaluation of numerical tools for fractional calculus and fractional order controls" *Int. J. Control.,* Vol. 90, 2017, pp. 1165-1181.

[17] J. F. Kelly and R. J. Mcgough, "Approximate analytical time-domain Green's functions for the Caputo fractional wave equation," *The Journal of the Acoustical Society of America,* Vol. 140, Nol. 2, 2016, pp. 1039-1047.

[18] K. Diethelm and N. J. Ford, "Analysis of fractional differential equations," *Journal of Mathematical Analysis and Applications,* Vol. 265, No. 2, 2002, pp. 229-248.

[19] H. Jafari and V. Daftardar-Gejji, "Solving linear and nonlinear fractional diffusion and wave equations by Adomian decomposition," *Applied Mathematics and Computation,* Vol. 180, No. 2, 2006, pp. 488-497.

[20] W. Rudin, *"Real and Complex Analysis, McGraw Hill.,"* Boston, 1987.

[21] M. Abramowitz and I. A. Stegun, *"Handbook of Mathematical Functions,"* New York, USA, 1972.

[22] H. M. Srivastava, "Fractional-order derivatives and integrals: Introductory overview and recent development," *Kyungpook Mathematical Journal*, 2020; Vol. 60, No. 1, 2020, pp. 73-116.

[23] F. W. J. Olver, D. W. Lozier, R. F. Boisvert and C. W. Clark, *"NIST Handbook of Mathematical Functions,"* Cambridge University Press, New York, USA, 2010.

[24] F. A. S. Salem and H. Irmak, "A special note on the error functions defined in certain domains of the complex plane and some of their implication," *WSEAS Transactions on Applied and Theoretical Mechanics,* Vol. 18, 2023, pp. 94-101, https://doi.org/10.37394/232011.2023.18.10

[25] H. Irmak, P. Agarwal and R. P. Agarwal, "The complex error functions and various extensive results together with implications

pertaining to certain special functions," *Turkish Journal of Mathematics*, Vol. 46, No. 2, 2022, pp. 662-667.

[26] G. Samko, A. A. Kılbas and O. I. Marichev, *Fractional Integrals and Derivatives: Theory and Applications*, Gordon and Breach, Yverdon, 1993.

[27] S. Owa, "On the distortion theorems. I," *Kyungpook Mathematical Journal.*, Vol. 18, 1978, pp. 53-59.

[28] I. N. Sneddon, *"The use of Integral Transforms,"* MacGraw-Hill, International Editions, New York, 1972.

[29] D. W. Brzeziński and P. Ostalczyk, "Numerical calculations accuracy comparison of the inverse Laplace transform algorithms for solutions of fractional order differential equations," *Nonlinear Dynamics*, Vol. 84, No. 1, 2016, pp. 65-77.

[30] R. Beals and R. Wong, "*Special Functions: A Graduate Text,"* Cambridge University Press, 2010.

[31] P. Henrici, *"Applied and Computational Complex Analysis,"* Wiley, New York, 1977.

[32] B. Davies, *"Integral Transforms and Their Applications,"* Springer, 2001.

[33] N. N. Lebedev, *"Special Functions and Their Applications,"* translated by R. A. Silverman, New Jersey, Prentice-Hall Inc, 1965.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors contributed to the present research, at all stages from the formulation of the problem to the final findings and solutions.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This research received no external funding.

Conflict of Interest

The authors have no conflict of interest to declare.