Nonlinear Stability Analysis of Stationary Solutions for a Special Class of Reaction-Diffusion Systems with Respect to Small Perturbations

QINGXIA LI[1], XINYAO YANG[2], ZIYAN ZHANG[1]
[1]Department of Mathematics and Computer Science, Fisk University
Nashville, Tennessee 37208,
USA
[2]Department of Applied Mathematics, Xi'an Jiaotong Liverpool University
Suzhou, Jiangsu 215123,
CHINA

Abstract: We prove that the stationary solution of a class of reaction-diffusion systems is stable in the intersection of the Sobolev space $H^1(\mathbb{R})$ and an exponentially weighted space $H^1_\alpha(\mathbb{R})$. Particular attention is given to a special case, the combustion model. The stationary solution considered here is the end state of the traveling front associated with the system, and thus the present result complements recent work by A. Ghazaryan, Y. Latushkin and S. Schecter, where the stability of the traveling fronts was investigated.

Key-Words: Dynamical Systems, Systems Theory, Reaction-Diffusion systems; Traveling waves; Stationary solutions; Essential spectrum; Exponential weight; Nonlinear stability.

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1 Introduction

The theory of reaction-diffusion equations emerged in the first half of the last century and has been influenced by various applications such as thermal explosions and the propagation of chemical waves. It brings together the theory of heat conduction and mass diffusion on the one hand and has a wide range of applications to chemical and biological kinetics on the other. Specifically, in the late 1930s, Kolmogorov-Petrovskii-Piskunov [6] and Fischer [3] proposed reaction-diffusion waves; subsequently, Zeldowitsch and Frank-Kamenetzki [10] studied them in conjunction with combustion theory. In this type of study, the reaction-diffusion equation

$$\partial_t u = \partial_x^2 u + F(u) \qquad (1)$$

is considered over the infinite domain $-\infty < x < \infty$, and u can be either the temperature or the concentration of the reactant. A traveling wave is a solution of the system (1) of the form $u(x,t) = w(x - ct)$, where c is a constant, i.e., the speed of the wave. This type of solution propagates with a constant speed and a certain shape and describe the asymptotic behavior of the system. The existence, stability and bifurcation problems of traveling waves are associated with many applications and mathematical models and have therefore been studied intensively in recent decades. In general, when analyzing the reaction-diffusion equation in unbounded domains, the invertibility of the limiting operator is required. This condition implies that the essential spectrum does not contain the origin, so we often need to study the essential spectrum of the linear operators of the system. In the practical application of this technique, especially in the study of some models in combustion and chemical kinetics, one often finds that the essential spectrum may contain the origin. In such cases, many conventional methods and theories are no longer applicable.

In this paper we will illustrate how to deal with this case in terms of the stability of the stationary solution of reaction-diffusion waves in one-dimensional space of a special class, and lead to some questions worth investigating. We will introduce the settings and some definitions in Section 2 and study the spectrum of the operator obtained by linearizing the equation with respect to the stationary solution in Section 3.1. Section 3.2 focuses on the nonlinear terms in the system and some nonlinear estimates needed to prove the main theorem. The proof of the stability of the stationary solution is given in Section 3.3, see Theorem 3.16. Finally, in Section 4 we give a generalization of the type of reaction-diffusion systems considered in [4,5].

2 Problem Formulation

We first briefly introduce a combustion model in $\mathbb{R}^d$ that includes two equations:

$$\begin{cases} u_{1t} = \Delta_x u_1 + u_2 g(u_1),\ u_1, u_2 \in \mathbb{R}, \\ u_{2t} = \epsilon \Delta_x u_2 - \kappa u_2 g(u_1),\ \boldsymbol{x} \in \mathbb{R}^d, \end{cases} \tag{2}$$

where the parameters ϵ and κ satisfy $0 \leq \epsilon \leq 1$ and $\kappa > 0$, and $g(\cdot)$ in the nonlinear terms is taken in the form of Arrihenius exponential:

$$g(u_1) = \begin{cases} e^{-\frac{1}{u_1}},\ if\ u_1 > 0; \\ 0 \quad ,\ if\ u_1 \leq 0. \end{cases} \tag{3}$$

Let $\boldsymbol{u} = \begin{pmatrix} u_1 \\ u_2 \end{pmatrix}$ and $f(\boldsymbol{u}) = \begin{pmatrix} f_1(u_1, u_2) \\ f_2(u_1, u_2) \end{pmatrix} = \begin{pmatrix} u_2 g(u_1) \\ -\kappa u_2 g(u_1) \end{pmatrix}$, then the system (2) can be rewritten in the vector form as

$$\boldsymbol{u}_t(t, \boldsymbol{x}) = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \Delta_x \boldsymbol{u}(t, \boldsymbol{x}) + f\big(\boldsymbol{u}(t, \boldsymbol{x})\big). \tag{4}$$

Given a fixed vector $\boldsymbol{e} \in \mathbb{R}^d$, the corresponding system written in the moving coordinate frame $\boldsymbol{x} + ct\boldsymbol{e}$ can be reduced to the equation

$$\mathbf{w}_t = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \Delta_{\mathbf{x}} \mathbf{w} + c(\mathbf{e} \cdot \nabla_{\mathbf{x}})\mathbf{w} + f(\mathbf{w}), \mathbf{w} \in \mathbb{R}^2. \tag{5}$$

We are concerned with the traveling wave solution of (5). In general, considering the practical model, such reaction-diffusion waves will approach the stationary states $\boldsymbol{u}_-$ and $\boldsymbol{u}_+$ exponentially as $z = \boldsymbol{x} \cdot \boldsymbol{e}$ tends to infinity, that is, there exist constants $K > 0$ and $\omega_- < 0 < \omega_+$ such that $\|\mathbf{w} - \boldsymbol{u}_-\| \leq \boldsymbol{K} e^{-\omega_- z}$ for $z \leq 0$ and $\|\mathbf{w} - \boldsymbol{u}_+\| \leq K e^{-\omega_+ z}$ for $z \geq 0$. The stability of the stationary solution with respect to small perturbations is usually determined by the spectrum of the linear operator. The situation is more complex in the case of traveling waves, because these are families of solutions, and the corresponding linear operator will have a zero eigenvalue. In this case, the discussion may involve a transfer of stability, i.e., the convergence of the solution of the non-stationary problem to a stationary solution in the family of solutions.

It is obvious that the system (5) has two types of stationary solutions: one when $u_1(\boldsymbol{x})$ is equal to a real constant and $u_2(\boldsymbol{x}) = 0$, and the other when $u_1(\boldsymbol{x}) = 0$ and $u_2(\boldsymbol{x})$ is equal to a real constant. In particular, we can choose $u_1 = 1/\kappa$, $u_2 = 0$, which is the state corresponding to completely burned reactants, and $u_1 = 0, u_2 = 1$, which corresponds to unburned substances. In other words, we choose $\boldsymbol{u}_- = (1/\kappa, 0)$ and $\boldsymbol{u}_+ = (0,1)$, see [4] to see why $\boldsymbol{u}_-$ and $\boldsymbol{u}_+$ are chosen this way. The one dimensional gasless combustion model of a solid fuel described by the system

$$\begin{aligned} \partial_t u_1 &= \partial_{xx} u_1 + u_2 g(u_1) \\ \partial_t u_2 &= -\kappa u_2 g(u_1), x \in \mathbb{R} \end{aligned} \tag{6}$$

has been studied in detail in [4], in which $\kappa > 0$, u_1 is the temperature, u_2 represents the concentration of unburned fuel. The authors of [4] investigated a traveling wave solution $(u_1, u_2)(\xi), \xi = x - ct$ for $c > 0$ where c is the speed of the front. Furthermore, $(u_1, u_2)(\xi)$ approaches the end states exponentially. However, the traveling wave is not spectrally stable in $H^1(\mathbb{R})$. The authors introduced a weight function $e^{\alpha\xi}$, where α is positive and small, such that the perturbation of the traveling wave belonging to this weighted space approaches 0 exponentially near the right end state, that is, as $\xi \to \infty$. In the weighted space, the nonlinear terms in (6) do not yield a locally Lipschitz mapping. To prove the stability of the traveling wave, the authors proved that perturbations of the traveling wave that are small in both the weighted norm and the unweighted norm will decay exponentially to the traveling wave in the weighted norm. As we will see in what follows, the study of the stability of the left end state of the front encounters similar difficulties.

We will now consider a time-independent solution of (5) of the form $\phi(z)$, where we assume that the function ϕ depends only on the scalar variable $z = \boldsymbol{x} \cdot \boldsymbol{e}$. We can perturb the function ϕ by either adding a function that depends only on z, that is, by considering the solution $\mathbf{w}(t, \boldsymbol{x})$ of (5) with initial condition

$$\mathbf{w}(0, \boldsymbol{x}) = \phi(z) + \mathbf{v}(0, z) \tag{7}$$

with some $\mathbf{v}: \mathbb{R} \times \mathbb{R} \to \mathbb{R}^2$ from an appropriate function space; or by adding a function that depends on $\boldsymbol{x}$, that is, consider the solution $\mathbf{w}(t, \boldsymbol{x})$ of (5) with the initial condition $\mathbf{w}(0, \mathbf{x}) = \phi(z) + \mathbf{v}(0, \mathbf{x})$ with some $\mathbf{v}: \mathbb{R} \times \mathbb{R}^d \to \mathbb{R}^2$ from an appropriate function space. The two types of perturbations lead to the spectral analysis of two different operators acting on $H^1(\mathbb{R})^2$ or $H^k(\mathbb{R}^{\mathrm{d}})^2$ respectively. We have already studied the second type of perturbation in [7]. In this paper we will study the first type as described in (7), noting that a similar approach as in [5] is mainly used here.

In particular, the study is related to the essential spectrum of the linear operator of the system. If the essential spectrum is in the right half-plane, it can be shifted to the left half-plane by introducing some exponentially weighted space. Given a real parameter α, we shall say that $\gamma_\alpha: \mathbb{R} \mapsto \mathbb{R}$ is a weight function of class α if $\gamma_\alpha(z) = e^{\alpha z}$ for $z \in \mathbb{R}$. The weighted space with the weight function γ_α is defined as

$$H_\alpha^1(\mathbb{R}) = \{u \in H_{loc}^1(\mathbb{R}): \gamma_\alpha(\cdot)u(\cdot) \in H^1(\mathbb{R})\}. \quad (8)$$

We denote the norm of u on the unweighted space $H^1(\mathbb{R})$ by $\|\cdot\|_0$ and the norm on the weighted space $H_\alpha^1(\mathbb{R})$ by $\| u \|_\alpha = \| \gamma_\alpha u \|_0$.

Since we will use only perturbations of the first type, we need a solution of the form $\mathbf{w}(t,z) = \mathbf{u}_- + \mathbf{v}(t,z)$, where $\mathbf{v}(t,\cdot)$ belongs to an appropriate space of functions on $\mathbb{R}$. With this notation we will have the following equation for the perturbation $\mathbf{v}(t,z)$:

$$\mathbf{v}_t = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \mathbf{v}_{zz} + c\mathbf{v}_z + f(\mathbf{v} + \mathbf{u}_-). \quad (9)$$

Introduction of the nonlinear term

$$H(\mathbf{v}) = f(\mathbf{u}_- + \mathbf{v}) - f(\mathbf{u}_-) - \partial_{\mathbf{u}} f(\mathbf{u}_-)\mathbf{v},$$

the equation (9) can be rewritten as follows:

$$\mathbf{v}_t = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \mathbf{v}_{zz} + c\mathbf{v}_z + \partial_{\mathbf{u}} f(\mathbf{u}_-)\mathbf{v} + H(\mathbf{v}). \quad (10)$$

Since

$$\partial_{\mathbf{u}} f(\mathbf{u}_-) = \begin{pmatrix} \frac{u_2}{u_1^2} e^{-\frac{1}{u_1}} & e^{-\frac{1}{u_1}} \\ -\kappa \frac{u_2}{u_1^2} e^{-\frac{1}{u_1}} & -\kappa e^{-\frac{1}{u_1}} \end{pmatrix}_{\mathbf{u}=\mathbf{u}_-} = \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix},$$

we therefore have

$$\mathbf{v}_t = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \mathbf{v}_{zz} + c\mathbf{v}_z + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix} \mathbf{v} + H(\mathbf{v}).$$

We now define the linear differential expression L with constant coefficients by

$$L = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \partial_{zz} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} c\, \partial_z + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix}. \quad (11)$$

A major difficulty is that the nonlinear term in (10) does not give a locally Lipschitz mapping on the weighted space $H_\alpha^1(\mathbb{R})$. To fix this problem, we introduce a new space:

$$\mathcal{E} := H^1(\mathbb{R}) \cap H_\alpha^1(\mathbb{R}). \quad (12)$$

with $\| u \|_{\mathcal{E}} = max\{\| u \|_0, \| u \|_\alpha\}$.

3 Stability of the Left End State

In this section, we will consider perturbing the right end state $\mathbf{u}_-$ of (5) with the perturbation as described in (7) and investigate the stability of the end state.

3.1 Spectral of the Linear Operators

To determine the stability of the perturbation as in (10), we need spectral information about the linear operator associated with (11). Consider the system of differential expressions L given by (11). We will now define several differential operators associated with L.

We define the linear operator $\mathcal{L}$ on $H^1(\mathbb{R})^2$ by the formula $\mathbf{u} \to L\mathbf{u}$ and the domain of $\mathcal{L}$ as the set of (u_1, u_2) where $u_1, u_2 \in H^3(\mathbb{R})$. For the space $L^2(\mathbb{R})$, the domain of $\mathcal{L}$ in $L^2(\mathbb{R})^2$ is the set of (u_1, u_2) where $u_1, u_2 \in H^2(\mathbb{R})$.

We will show below that the spectrum of $\mathcal{L}$ touches the imaginary axis so that the equilibrium solution is not spectrally stable in $H^1(\mathbb{R})^2$. A way out of this problem is then to use a weighted space $H_\alpha^1(\mathbb{R})^2$.

We define the operator $\mathcal{L}_\alpha$ on $H_\alpha^1(\mathbb{R})^2$ as the linear operator given by the formula $\mathbf{u} \to L\mathbf{u}$, and the domain of $\mathcal{L}_\alpha$ is the set

$$\{(u_1, u_2): \gamma_\alpha(\cdot)u_1(\cdot), \gamma_\alpha(\cdot)u_2(\cdot) \in H^3(\mathbb{R})\},$$

see formula (8). Similarly, we define the weighted space $L_\alpha^2(\mathbb{R}) := \{u: \gamma_\alpha(\cdot)u(\cdot) \in L^2(\mathbb{R})\}$, the domain of $\mathcal{L}_\alpha$ on $L_\alpha^2(\mathbb{R})^2$ is the set:

$$\{(u_1, u_2): \gamma_\alpha(\cdot)u_1(\cdot), \gamma_\alpha(\cdot)u_2(\cdot) \in H^2(\mathbb{R})\}.$$

We denote by $\mathcal{L}_{\mathcal{E}}$ the linear operator on $\mathcal{E}^2$ given by $\mathbf{u} \to L\mathbf{u}$ where the domain of $\mathcal{L}_{\mathcal{E}}$ is the set of $\mathbf{u}$ on $\mathcal{E}^2$ satisfying $\mathbf{u} \in \mathrm{dom}(\mathcal{L}) \cap \mathrm{dom}(\mathcal{L}_\alpha)$, where $\mathrm{dom}(\mathcal{L})$ and $\mathrm{dom}(\mathcal{L}_\alpha)$ are the respective domains defined above.

In the remaining part of this subsection, we will collect several elementary facts about the spectrum of the differential operators $\mathcal{L}$ and $\mathcal{L}_\alpha$ on the respective spaces. We recall here that for a general closed densely defined operator $\mathcal{T}$, the resolvent set $\rho(\mathcal{T})$ is the set of $\lambda \in \mathbb{C}$ such that $\mathcal{T} - \lambda I$ has a bounded inverse. The complement of $\rho(\mathcal{T})$ is the spectrum $\sigma(\mathcal{T})$. It is the union of the discrete spectrum $\sigma_d(\mathcal{T})$, which is the set of isolated points in $\sigma(\mathcal{T})$ that are eigenvalues of $\mathcal{T}$ of finite algebraic multiplicity, and the essential spectrum $\sigma_{ess}(\mathcal{T})$, which is the rest. We will use the Fourier transform to find $\sigma(\mathcal{L})$ on $L^2(\mathbb{R})^2$. First, we notice that the operator $\mathcal{L}$ on $L^2(\mathbb{R})^2$ is similar to the operator of multiplication on $L^2(\mathbb{R})^2$ by the matrix-valued function $M(\theta)$, where

$$M(\theta) = -\begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \theta^2 + i\theta c I + \partial_{\mathbf{u}} f(\mathbf{u}_-), \theta \in \mathbb{R}, (13)$$

see e.g. [2. Section 6.5]. The spectrum of $\mathcal{L}$ on $L^2(\mathbb{R})^2$ is the closure of the union over $\theta \in \mathbb{R}$ of the spectra of the matrices $M(\theta)$. Hence the spectrum of $\mathcal{L}$ is equal to the closure of the set of $\lambda \in \mathbb{C}$ for which there exists $\theta \in \mathbb{R}$ such that

$$\det(M(\theta)-\lambda I)=\det(-\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix}\theta^2 +(i\theta c-\lambda)I+\partial_{\mathbf{u}}f(\mathbf{u}_-))=0.$$

It is a collection of curves of the form $\lambda=\lambda_k(\theta)$, where $\lambda_k(\theta)$ are the eigenvalues of the matrices $M(\theta)$.

The spectrum of $\mathcal{L}$ on $L^2(\mathbb{R})^2$ is equal to its spectrum on $H^1(\mathbb{R})^2$, which is proved in the following lemma.

Lemma 3.1. *The linear operator $\mathcal{L}$ with constant coefficients associated with the differential expression L in (11) has the same spectrum on $L^2(\mathbb{R})^2$ and $H^1(\mathbb{R})^2$.*

Proof. We will denote the operator associated with L by $\mathcal{L}_{L^2}$ on $L^2(\mathbb{R})^2$ and by $\mathcal{L}_{H^1}$ on $H^1(\mathbb{R})^2$. Recall that ∂_{z,L^2} has the domain $H^1(\mathbb{R})$ and spectrum $i\mathbb{R}$. Therefore, the operator

$$\mathcal{D}=\begin{pmatrix}\partial_{z,L^2}+\mathcal{I} & 0\\ 0 & \partial_{z,L^2}+\mathcal{I}\end{pmatrix}:$$
$$H^1(\mathbb{R})\times H^1(\mathbb{R})\mapsto L^2(\mathbb{R})\times L^2(\mathbb{R})$$

is an isomorphism. Using the identity $\mathcal{D}\mathcal{L}_{H^1}\mathbf{v}=\mathcal{L}_{L^2}\mathcal{D}\mathbf{v}$ for all $\mathbf{v}\in\operatorname{dom}\mathcal{L}_{H^1}=H^3(\mathbb{R})^2$, we get $\mathcal{D}\mathcal{L}_{H^1}\mathcal{D}^{-1}=\mathcal{L}_{L^2}$. Thus, we can conclude that $\sigma(\mathcal{L}_{H^1})=\sigma(\mathcal{D}\mathcal{L}_{L^2}\mathcal{D}^{-1})=\sigma(\mathcal{L}_{L^2})$ as claimed. □

By Lemma 3.1 and the preceding discussion, for the operator $\mathcal{L}$ associated with the differential expression

$$L=\begin{pmatrix}\partial_{zz}+c\,\partial_z & e^{-\kappa}\\ 0 & \epsilon\,\partial_{zz}+c\,\partial_z-\kappa e^{-\kappa}\end{pmatrix},\quad (14)$$

the spectrum of $\mathcal{L}$ on $L^2(\mathbb{R})^2$ and $H^1(\mathbb{R})^2$ is

$$\sigma(\mathcal{L})=\cup_{\theta\in\mathbb{R}}\,\sigma\begin{pmatrix}-\theta^2+ci\theta & e^{-\kappa}\\ 0 & -\epsilon\theta^2+ci\theta-\kappa e^{-\kappa}\end{pmatrix}$$
$$=\cup_{\theta\in\mathbb{R}}(-\theta^2+ci\theta)\bigcup\cup_{\theta\in\mathbb{R}}(-\epsilon\theta^2+ci\theta-\kappa e^{-\kappa}).\quad (15)$$

Hence, the spectrum of $\mathcal{L}$ on $L^2(\mathbb{R})^2$ is the union of the two curves $\lambda_1=-\theta^2+ci\theta$ and $\lambda_2=\epsilon\,\partial_{zz}+c\,\partial_z-\kappa e^{-\kappa}$ where $\theta\in\mathbb{R}$, therefore $sup\{Re\,\lambda:\lambda\in\sigma(\mathcal{L})\}=0$. Thus the spectrum of $\mathcal{L}$ on $L^2(\mathbb{R})^2$ and $H^1(\mathbb{R})^2$ touches the imaginary axis.

Next, we will tackle $\sigma(\mathcal{L}_\alpha)$ on $H^1_\alpha(\mathbb{R})^2$, which can be described as follows. Let $\mathcal{E}_0$ be $L^2(\mathbb{R})$ or $H^1(\mathbb{R})$ and $\mathcal{E}_\alpha=\{u:\gamma_\alpha(z)u(z)\in\mathcal{E}_0\}$. The linear operator $\mathcal{M}$ defined by $\mathcal{M}\mathbf{u}=\gamma_\alpha\mathbf{u}$ is an isomorphism from $\mathcal{E}^2_\alpha$ to $\mathcal{E}^2_0$. Define the linear operator $\hat{\mathcal{L}}=\mathcal{M}\mathcal{L}_\alpha\mathcal{M}^{-1}$ on $\mathcal{E}^2_0$, with domain $H^2(\mathbb{R})^2$ if $\mathcal{E}^2_0=L^2(\mathbb{R})^2$, or domain $H^3(\mathbb{R})^2$ if $\mathcal{E}^2_0=H^1(\mathbb{R})^2$. It is therefore similar to $\mathcal{L}_\alpha$ on $\mathcal{E}^2_\alpha$ and hence has the same spectrum.

Assume that (v_1,v_2) belongs to the weighted space with the weight function γ_α. It follows that $(v_1,v_2)=\gamma_\alpha^{-1}(\tilde{v}_1,\tilde{v}_2)$ with $\tilde{\mathbf{v}}=(\tilde{v}_1,\tilde{v}_2)\in L^2(\mathbb{R})^2$. By substituting into the formula for $L\begin{pmatrix}v_1\\ v_2\end{pmatrix}$ and multiplying by γ_α and noticing that $\hat{\partial}_z\tilde{\mathbf{v}}=\gamma_\alpha\partial_z\gamma_\alpha^{-1}\tilde{\mathbf{v}}=\gamma_\alpha((\gamma_\alpha^{-1})'\tilde{\mathbf{v}}+\gamma_\alpha^{-1}\tilde{\mathbf{v}})=(\partial_z-\alpha)\tilde{\mathbf{v}}$, we can rewrite the linear differential expression

$$\hat{L}=\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix}(\partial_z-\alpha)^2+c(\partial_z-\alpha)+\begin{pmatrix}0 & e^{-\kappa}\\ 0 & -\kappa e^{-\kappa}\end{pmatrix}=L-2\alpha\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix}\partial_z+\alpha^2\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix}-c\alpha I.$$

Via the Fourier transfom, the operator $\hat{\mathcal{L}}$ on $L^2(\mathbb{R})^2$ is similar to the operator of multiplication on $L^2(\mathbb{R})^2$ by the matrix-valued function

$$N(\theta)=-\theta^2\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix}+i\theta(cI-2\alpha\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix}) +\alpha^2\begin{pmatrix}1 & 0\\ 0 & \epsilon\end{pmatrix}-c\alpha I+\begin{pmatrix}0 & e^{-\kappa}\\ 0 & -\kappa e^{-\kappa}\end{pmatrix}.$$

Hence the spectrum of $\hat{\mathcal{L}}$ on $L^2(\mathbb{R})^2$ equal to that of multiplication by N on $L^2(\mathbb{R})^2$.

Thus, we find that the spectrum of the operator $\hat{\mathcal{L}}$ is the union of the two curves $\lambda_1=-\theta^2+(c-2\alpha)\theta i+\alpha^2-c\alpha$ and $\lambda_2=-\epsilon\theta^2+(c-2\alpha\epsilon)\theta i+\epsilon\alpha^2-c\alpha-\kappa e^{-\kappa}$ for all $\theta\in\mathbb{R}$. Then

$$sup\{\mathrm{Re}\,\lambda:\lambda\in\sigma_{ess}(\mathcal{L}_\alpha)\}$$
$$=sup\{\mathrm{Re}\,\lambda:\lambda\in\sigma_{ess}(\hat{\mathcal{L}})\}$$
$$=max\{\alpha^2-c\alpha,\epsilon\alpha^2-c\alpha-\kappa e^{-\kappa}\}$$
$$=\alpha^2-c\alpha.$$

The linear operator $\mathcal{L}_\alpha$ is an operator with constant coefficients, so $\sigma(\mathcal{L}_\alpha)=\sigma_{ess}(\mathcal{L}_\alpha)$. We also have the following analogue of Lemma 3.1.

Lemma 3.2. *The linear operator $\mathcal{L}_\alpha$ associated with the differential expression, that is defined in (14), has the same spectrum on $L^2_\alpha(\mathbb{R})^2$ and $H^1_\alpha(\mathbb{R})^2$.*

For $\alpha\in(0,c/2)$, we will have $sup\{\mathrm{Re}\,\lambda:\lambda\in\sigma(\mathcal{L}_\alpha)\}<0$ so that the spectrum $\sigma(\mathcal{L}_\alpha)$ has been moved to the left of the imaginary axis. We summarize this result as the following proposition.

Proposition 3.3. *On the unweighted space $H^1(\mathbb{R})^2$, one has $sup\{Re\,\lambda:\lambda\in\sigma(\mathcal{L})\}=0$, the spectrum of $\mathcal{L}$ will touch the imaginary axis. On the weighted space $H^1_\alpha(\mathbb{R})^2$, if $0<\alpha<c/2$, then the spectrum of $\mathcal{L}_\alpha$ will be bounded away from the imaginary axis and $sup\{Re\,\lambda:\lambda\in\sigma(\mathcal{L}_\alpha)\}<-\nu$ for some $\nu>0$.*

In the system (10) we have the following triangular structure,

$$\mathbf{v}_t=\begin{pmatrix}\partial_{zz}+c\,\partial_z & e^{-\kappa}\\ 0 & \epsilon\,\partial_{zz}+c\,\partial_z-\kappa e^{-\kappa}\end{pmatrix}\mathbf{v}+H(\mathbf{v}).$$

Let

$$L_1 = \partial_{zz} + c\,\partial_z, \tag{16}$$
$$L_2 = \epsilon\,\partial_{zz} + c\,\partial_z - \kappa e^{-\kappa}, \tag{17}$$

and for $i = 1,2$, let $\mathcal{L}_i$ be the operators on $H^1(\mathbb{R})$ defined by $v_i \mapsto L_i v_i$, the domain of $\mathcal{L}_i$ on $H^1(\mathbb{R})$ is the set of v_i where $v_i \in H^3(\mathbb{R})$.

Lemma 3.4. *Consider the operators $\mathcal{L}$, $\mathcal{L}_1$ and $\mathcal{L}_2$ associated with the differential expressions (14), (16), and (17), respectively,*

1) *The operator $\mathcal{L}_1$ generates a bounded semigroup on $H^1(\mathbb{R})$;*
2) *the operator $\mathcal{L}_2$ on $H^1(\mathbb{R})$ satisfies $sup\{Re\,\lambda: \lambda \in \sigma(\mathcal{L}_2)\} < 0$;*
3) *the following is true on $H^1(\mathbb{R})$:*
   (a) *$sup\{Re\,\lambda: \lambda \in \sigma(\mathcal{L}_1)\} \leq 0$;*
   (b) *$sup\{Re\lambda: \lambda \in \sigma(\mathcal{L})\} \leq 0$;*
   (c) *there exist $\rho > 0$ and $K > 0$ such that $||e^{t\mathcal{L}_2}||_{H^1(\mathbb{R})\to H^1(\mathbb{R})} \leq Ke^{-\rho t}$ for $t \geq 0$.*

Proof. We claim that the semigroups generated by the operators $\mathcal{L}_i$, $i = 1,2$, on $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ are similar (this gives yet another way to prove that $\mathcal{L}_i$ has the same spectrum on $L^2(\mathbb{R})$ and $H^1(\mathbb{R})$ for $i = 1,2$). We denote $\mathcal{L}_i$ on $L^2(\mathbb{R})$ as $\mathcal{L}_{iL^2}$ and $\mathcal{L}_i$ on $H^1(\mathbb{R})$ as $\mathcal{L}_{iH^1}$.

Recall that the Fourier transform is an isomorphism of $H^1(\mathbb{R})$ onto $L^2_m(\mathbb{R})$, where the weight function is $m(\theta) = (1 + |\theta|^2)^{1/2}$ for $\theta \in \mathbb{R}$. The operator of multiplication by the function $m(\theta)$ is an isomorphism of $L^2_m(\mathbb{R})$ onto $L^2(\mathbb{R})$. Under the Fourier transform followed by this isomorphism of $H^1(\mathbb{R})$ onto $L^2(\mathbb{R})$, the operator of differentiation on $H^1(\mathbb{R})$ is similar to the operator of multiplication by $i\theta$ on $L^2(\mathbb{R})$. Using this, we have $m\mathcal{F}_1\mathcal{L}_{iH^1} = M(\theta)m\mathcal{F}_1$, where $M(\theta)$ is defined in (13). The operator of multiplication by $i\theta$ on $L^2(\mathbb{R})$ is similar to the operator of differentiation on $L^2(\mathbb{R})$ via the Fourier transform, and thus we have $\mathcal{F}_2\mathcal{L}_{iL^2} = M(\theta)\mathcal{F}_2$. It follows that

$$\begin{aligned}\mathcal{L}_{iH^1} &= (m\mathcal{F}_1)^{-1}M(\theta)m\mathcal{F}_1\\ &= (m\mathcal{F}_1)^{-1}(\mathcal{F}_2\mathcal{L}_{iL^2}\mathcal{F}_2^{-1})(m\mathcal{F}_1),\end{aligned}$$

and thus the operators on $H^1(\mathbb{R})$ and $L^2(\mathbb{R})$ associated with the same constant-coefficient differential expression are similar. Therefore the semigroups they generate are similar, proving the claim.

The operator $\mathcal{L}_1$ generates a bounded semigroup on $L^2(\mathbb{R})$ by Proposition A.1(1) of [6]. Thus, 1) is proved because $\mathcal{L}_1$ on $H^1(\mathbb{R})$ is similar to $\mathcal{L}_1$ on $L^2(\mathbb{R})$.

Using the Fourier transform, we can find that the spectrum of $\mathcal{L}_1$ on $L^2(\mathbb{R})$ is the curve $\lambda_1 = -\theta^2 + ci\theta$ and the spectrum of $\mathcal{L}_2$ on $L^2(\mathbb{R})$ is the curve $\lambda_2 = -\epsilon\theta^2 + ci\theta - \kappa e^{-\kappa}$. Thus $sup\{Re\,\lambda: \lambda \in \sigma(\mathcal{L}_1)\} \leq 0$ and $sup\{Re\,\lambda: \lambda \in \sigma(\mathcal{L}_2)\} < 0$ on $L^2(\mathbb{R})$. It is also true on $H^1(\mathbb{R})$, proving statements 2) and 3) (a), (b).

Statement 3)(c) is a direct consequence of 2), see [5, Lemma 3.13]. □

3.2 Lipschitz Property of the Nonlinear Term

In this subsection, we will mainly focus on the nonlinear term $H(\mathbf{v})$ in (10) and will show that the nonlinear term yields a locally Lipschitz mapping on the intersection space $\mathcal{E}$. This exposition is quite elementary by nature, and we present it here because it is a necessary condition for proving the existence of solutions of (10). In particular, we substitute $\mathbf{u}_- = (1/\kappa, 0)$ into the nonlinear term $H(\mathbf{v})$ to obtain

$$\begin{aligned}H(\mathbf{v}) &= f(\begin{pmatrix}1/\kappa\\0\end{pmatrix} + \begin{pmatrix}v_1\\v_2\end{pmatrix}) - \begin{pmatrix}0 & e^{-\kappa}\\0 & -\kappa e^{-\kappa}\end{pmatrix}\begin{pmatrix}v_1\\v_2\end{pmatrix}\\ &= \begin{pmatrix}v_2 e^{-\frac{1}{v_1+1/\kappa}}\\ -\kappa v_2 e^{-\frac{1}{v_1+1/\kappa}}\end{pmatrix} - \begin{pmatrix}v_2 e^{-\kappa}\\ -\kappa v_2 e^{-\kappa}\end{pmatrix}\\ &= \begin{pmatrix}v_2(e^{-\frac{1}{v_1+1/\kappa}} - e^{-\kappa})\\ -\kappa v_2(e^{-\frac{1}{v_1+1/\kappa}} - e^{-\kappa})\end{pmatrix}.\end{aligned}$$

We introduce the notation $\mathbf{k} = \begin{pmatrix}1\\-\kappa\end{pmatrix}$, then $H(\mathbf{v})$ can be written as

$$H(\mathbf{v}) = \mathbf{k}(g(\tfrac{1}{\kappa} + v_1) - g(\tfrac{1}{\kappa}))v_2, \tag{18}$$

where $g(\cdot)$ is defined as in the equation (3).

In order to prove that $H(\mathbf{v})$ is a locally Lipschitz mapping on an appropriate space, we will use below the inclusion $g(u) \in C^\infty(\mathbb{R})$ when $u \in \mathbb{R}$. To prove this inclusion, we first show that $\lim_{u\to0^+} g^{(n)}(u) = 0$ for $n \in \mathbb{N}$. Indeed, by L'Hospital's rule for $n \in \mathbb{N}$,

$$\lim_{u\to0^+} u^{-n}e^{-1/u} = \lim_{x\to\infty} x^n e^{-x} = 0. \tag{19}$$

On the other hand, if $u > 0$ then $g(u) = e^{-\frac{1}{u}}$ and it follows that

$$\begin{aligned}g'(u) &= e^{-\frac{1}{u}}u^{-2},\\ g''(u) &= e^{-\frac{1}{u}}(u^{-4} - 2u^{-3}), \cdots\\ g^{(n)}(u) &= e^{-\frac{1}{u}}(u^{-2n} + c_{-2n+1}u^{-2n+1} + \cdots\\ &\quad + c_{-n-1}u^{-n-1})\end{aligned}$$

are all continuous functions for $u > 0$. Using (19), we can conclude that $g^{(n)}(u)$ approaches 0 as $u \to 0^+$ for all $n \in \mathbb{N}$ and thus $g^{(n)}(\cdot)$ is continuous for all u. The required inclusion $g(u) \in \mathbb{C}^\infty(\mathbb{R})$follows.

We recall notation (12), that is, $\mathcal{E} = H^1(\mathbb{R}) \cap H^1_\alpha(\mathbb{R})$ with the norm $\| u \|_{\mathcal{E}} = max\{\| u \|_0, \| u \|_\alpha\}$. In order to prove the Lipschitz property of $H(\mathbf{v})$ on $\mathcal{E}$, we will also need the following elementary proposition.

Proposition 3.5.

(1) If $u, v \in H^1(\mathbb{R})$, then $uv \in H^1(\mathbb{R})$; furthermore, there exists a constant $C > 0$ such that $||uv||_0 \le C||u||_0||v||_0$.

(2) If $u, v \in \mathcal{E}$, then $uv \in H^1_\alpha(\mathbb{R})$; furthermore, there exists a constant $C > 0$ such that $||uv||_\alpha \le C||u||_0||v||_\alpha$.

(3) If $u, v \in \mathcal{E}$, then $uv \in \mathcal{E}$; also there exists a constant $C > 0$ such that $||uv||_{\mathcal{E}} \le C||u||_{\mathcal{E}}||v||_{\mathcal{E}}$.

Proof. Assertion (1) is a well-known result of Sobolev spaces, see [1, Theorem 5.23]. Assertion (2) can be proved by

$$||uv||_\alpha = ||\gamma_\alpha uv||_0 \le C||u||_0||\gamma_\alpha v||_0 = C||u||_0||v||_\alpha.$$

To show assertion (3), let $u, v \in \mathcal{E}$. Then by assertion (1),

$$||uv||_0 \le C||u||_0||v||_0 \le C||u||_{\mathcal{E}}||v||_{\mathcal{E}},$$

and by assertion (2), $||uv||_\alpha \le C||u||_0||v||_\alpha \le C||u||_{\mathcal{E}}||v||_{\mathcal{E}}$. Therefore $uv \in \mathcal{E}$ and $||uv||_{\mathcal{E}} \le C||u||_{\mathcal{E}}||v||_{\mathcal{E}}$. □

Let $\mathcal{U} \subset \mathbb{R}$ and let $C^0(\mathcal{U})$ denote the space of bounded C^0 functions $m: \mathcal{U} \to \mathbb{R}$ with the sup norm, which we now denote $|| \cdot ||_{C^0}$. More generally, let $C^k(\mathcal{U})$ denote the space of C^k functions $m: \mathcal{U} \to \mathbb{R}$ such that $m, \partial m, \cdots, \partial^k m$ are all bounded functions, with the C^k-norm:

$$||m||_{C^k} = ||m||_{C^0} + ||\,\partial m||_{C^0} + \cdots + ||\,\partial^k m||_{C^0}.$$

Proposition 3.6. *Let $m(\cdot) \in C^2(\mathbb{R})$. Then the formula $v(z) \mapsto m(v(z))$ defines mappings from $H^1(\mathbb{R})$ to $H^1(\mathbb{R})$ and from $\mathcal{E}$ to $\mathcal{E}$. The first is Lipschitz on any set of the form $\{v: ||v||_0 \le K\}$; the second is Lipschitz on any set of the form $\{v: ||v||_{\mathcal{E}} \le K\}$.*

Proof. We have

$$m(v(z) + \overline{v}(z)) - m(v(z)) = \int_0^1 \partial_v m(v(z) + t\overline{v}(z))dt\,\overline{v}(z). \tag{20}$$

Therefore, $\| m(v + \overline{v}) - m(v) \|_{L^2} \le ||m||_{C^1}||\overline{v}||_{L^2}$, and

$$||\gamma_\alpha(m(v + \overline{v}) - m(v))||_{L^2} \le ||m||_{C^1}||\gamma_\alpha\overline{v}||_{L^2}.$$

Also, differentiating the equation (20) we have

$$\begin{aligned}\partial_z(m(v(z) + \overline{v}(z)) - m(v(z))) &= \int_0^1 \partial_v^2 m(v(z) + t\overline{v}(z))(\partial_z v + t\,\partial_z\overline{v})dt\,\overline{v}(z) \\ &+ \int_0^1 \partial_v m(v(z) + t\overline{v}(z))dt\,\partial_z\overline{v}.\end{aligned}$$

Thus

$$\begin{aligned}\| \partial_z(m(v(z) + \overline{v}(z)) - m(v(z))) \|_{L^2} &\le ||m||_{C^2}||\,\partial_z v||_{L^2}||\overline{v}||_{L^\infty} \\ &+ \frac{1}{2}||m||_{C^2}||\,\partial_z\overline{v}||_{L^2}||\overline{v}||_{L^\infty} \\ &+ ||m||_{C^1}||\,\partial_z\overline{v}||_{L^2},\end{aligned}$$

and since $\overline{v} \in H^1(\mathbb{R}) \subset L^\infty(\mathbb{R})$ by the Sobolev embedding theorem, we have

$$\begin{aligned}||\,\partial_z(m(v(z) + \overline{v}(z)) - m(v(z)))||_{L^2} &\le ||m||_{C^2}||\,\partial_z v||_{L^2}C||\overline{v}||_{H^1} \\ &+ \frac{1}{2}||m||_{C^2}||\,\partial_z\overline{v}||_{L^2}C||\overline{v}||_{H^1} \\ &+ ||m||_{C^1}||\,\partial_z\overline{v}||_{L^2},\end{aligned}$$

similarly,

$$\begin{aligned}||\gamma_\alpha\,\partial_z(m(v(z) + \overline{v}(z)) - m(v(z)))||_{L^2} &\le ||m||_{C^2}||\,\partial_z v||_{L^2}C||\gamma_\alpha\overline{v}||_{H^1} \\ &+ \frac{1}{2}||m||_{C^2}||\,\partial_z\overline{v}||_{L^2}C||\gamma_\alpha\overline{v}||_{H^1} \\ &+ ||m||_{C^1}||\gamma_\alpha\,\partial_z\overline{v}||_{L^2}.\end{aligned}$$

If $||v||_0$ and $||v + \overline{v}||_0$ are both bounded by the constant K, then $||\overline{v}||_0 \le 2K$, and due to the equation (20) there exists a constant $C_K > 0$ depending on K, such that

$$||m(v(z) + \overline{v}(z)) - m(v(z))||_0 \le C_K||\overline{v}(z)||_0.$$

Similarly, if $||v||_{\mathcal{E}}, ||v + \overline{v}||_{\mathcal{E}}$ are bounded by the constant K, then $||\overline{v}||_{\mathcal{E}} \le 2K$,

$$||m(v(z) + \overline{v}(z)) - m(v(z))||_0 \le C_K||\overline{v}||_0$$

and

$$||m(v(z) + \overline{v}(z)) - m(v(z))||_\alpha \le C_K||\overline{v}||_\alpha,$$

thus

$$||m(v(z) + \overline{v}(z)) - m(v(z))||_{\mathcal{E}} \le C_K||\overline{v}||_{\mathcal{E}}.$$

□

Proposition 3.7. *Let $m(\cdot) \in C^2(\mathbb{R})$. Consider the formula*

$$(u(z), v(z)) \mapsto m(u(z))v(z). \tag{21}$$

(1) Formula (21) defines a mapping from $H^1(\mathbb{R})^2$ to $H^1(\mathbb{R})$ that is locally Lipschitz on any set of the form $\{(u, v): ||u||_0 + ||v||_0 \le K\}$.

(2) Formula (21) defines a mapping from $\mathcal{E}^2$ to $\mathcal{E}$ that is locally Lipschitz on any set of the form $\{(u, v): ||u||_{\mathcal{E}} + ||v||_{\mathcal{E}} \le K\}$.

Proof. See Proposition 5.6 in [4]. □

Proposition 3.8. *Let $m(v_1) = g(1/\kappa + v_1) - g(1/\kappa)$ with g from (3).*

(1) The formula $v_1(z) \mapsto m(v_1(z))$ defines a mapping from $H^1(\mathbb{R})$ to $H^1(\mathbb{R})$ that is Lipschitz on any set of the form $\{v_1: ||v_1||_0 \leq K\}$ and there is a constant $C_K > 0$ depending on K such that $||m(v_1)||_0 \leq C_K||v_1||_0$.

(2) If $\boldsymbol{v} = (v_1, v_2) \in H^1(\mathbb{R})^2$, and $H(\boldsymbol{v}) = \boldsymbol{k}(g(1/\kappa + v_1) - g(1/\kappa))$ is given by the equation (18), then $H(\boldsymbol{v})$ defines a mapping from $H^1(\mathbb{R})^2$ to $H^1(\mathbb{R})^2$ that is Lipschitz on any set of the form $\{\boldsymbol{v}: ||\boldsymbol{v}||_0 \leq K\}$ and $||H(\boldsymbol{v})||_0 \leq C_K \| v_1 \|_0 \| v_2 \|_0 \leq C_K||\boldsymbol{v}||_0^2$.

Proof. (1) The Lipschitz property follows from Proposition 3.6. Since the mapping is Lipschitz, and $m(0) = g(1/\kappa + 0) - g(1/\kappa) = 0$, we then have that $\| m(v_1) \|_0 \leqslant C_K \| v_1 \|_0$.

(2) Since $H(\mathbf{v}) = \mathbf{k}m(v_1)v_2$, assertion (2) follows from assertion (1) and Proposition 3.7. (2). ☐

Proposition 3.9. *Consider the formula*

$$\mathbf{v} = (v_1(z), v_2(z)) \mapsto H(v_1(z), v_2(z)), \quad (22)$$

where $H(\cdot)$ is given by the equation (18).

(1) If $\mathbf{v} \in \mathcal{E}^2$, then $H(\mathbf{v}) \in H^1_\alpha(\mathbb{R})^2$, also $H(\cdot)$ is Lipschitz on any set of the form $\{\mathbf{v}: ||\mathbf{v}||_\mathcal{E} \leq K\}$ and there exists a constant $C_K > 0$ depending on K such that $||H(\mathbf{v})||_\alpha \leq C_K||\mathbf{v}||_0||\mathbf{v}||_\alpha$.

(2) Formula (18) for $H(v_1, v_2)$ defines a mapping from $\mathcal{E}^2$ to $\mathcal{E}^2$ that is Lipschitz on any set of the form $\{\mathbf{v}: ||\mathbf{v}||_\mathcal{E} \leq K\}$ and $||H(\mathbf{v})||_\mathcal{E} \leq C_K||\mathbf{v}||^2_\mathcal{E}$.

Proof. To prove assertion (1), we use the following inequality:

$$\begin{aligned}||H(\mathbf{v})||_\alpha &= ||\gamma_\alpha H(\mathbf{v})||_0 \\ &= ||\gamma_\alpha \mathbf{k} v_2(g(v_1 + 1/\kappa) \\ &\quad - g(1/\kappa))||_0 \\ &\leq ||\mathbf{k}m(v_1(z))||_0||\gamma_\alpha v_2||_0 \\ &\leq C_K||v_1||_0||v_2||_\alpha \\ &\leq C_K||\mathbf{v}||_0||\mathbf{v}||_\alpha\end{aligned}$$

for some $C_K > 0$ depending on K. Assertion (2) can be proved similarly to Proposition 3.8. (2) by using Proposition 3.7. (2). ☐

3.3 Nonlinear Stability Analysis

In this subsection, we prove the stability of the right end state $\mathbf{u}_-$ of the system (5) on $\mathcal{E}^2$. The operator $\mathcal{L}_\mathcal{E}$ generates a strongly continuous semigroup on $\mathcal{E}^2$. The nonlinear term yields a locally Lipschitz mapping on $\mathcal{E}^2$ by Proposition 3.9. Therefore we can apply the following standard result.

Lemma 3.10. *Let $\mathcal{X}$ be a Banach space. Consider the system*

$$\mathbf{v}_t = \mathcal{L}\mathbf{v}(t) + H(\mathbf{v}(t)), t \geq 0,$$

where $H(\mathbf{v})$ is locally Lipschitz continuous in $\mathbf{v}$ and the operator $\mathcal{L}: dom(\mathcal{L}) \subset \mathcal{X} \mapsto \mathcal{X}$ generates a C_0 semigroup $T(t)$ on $\mathcal{X}$.

For any $\mathbf{v}^0 \in \mathcal{X}$ the system has a unique mild solution $\boldsymbol{v}$ with the initial value $\mathbf{v}^0$. The solution is defined for the time t in the maximal interval $0 \leq t < t_{max}(\mathbf{v}^0)$ where $0 < t_{max}(\mathbf{v}^0) \leq \infty$.

Proof. See [8, Theorem 6.1.4]. ☐

Next, we recall yet another standard fact. Consider a system of the form $\mathbf{v}_t = L\mathbf{v} + H(\mathbf{v}(t))$, where the operator $\mathcal{L}$ is defined by the formula $\mathbf{v} \mapsto L\mathbf{v}$ with the domain $dom(\mathcal{L}) \subset \mathcal{E}$ and generates a C_0-semigroup on $\mathcal{E}$, and $H(\cdot)$ is a locally Lipschitz mapping from $\mathcal{E}$ into $\mathcal{E}$.

Let E be given by

$$E = \{(\mathbf{v}^0, t) \in \mathcal{E} \times \mathbb{R}^+: 0 \leq t < t_{max}(\mathbf{v}^0)\};$$

so that the set E is open in $\mathcal{E} \times \mathbb{R}^+$, and the map $(\mathbf{v}^0, t) \mapsto \mathbf{v}$ from E to $\mathcal{E}$ is continuous. We have the following lemma.

Lemma 3.11. *For each $\delta > 0$, if $0 < \gamma < \delta$, then there exists T depending on γ and δ, with $0 < T \leq \infty$, such that the following holds: if $\mathbf{v}^0 \in \mathcal{E}$ satisfies*

$$||\mathbf{v}^0||_\mathcal{E} \leq \gamma \quad (23)$$

and $0 \leq t < T$, then the solution $\mathbf{v}(t) \in \mathcal{E}$ of the system (10) is defined and satisfies

$$||\mathbf{v}(t)||_\mathcal{E} \leq \delta. \quad (24)$$

Proof. The proof is the same as the proof in [9, Theorem 46.4]. ☐

If $\delta, \gamma > 0$ are fixed, let $T(\gamma, \delta)$ denote the supremum of all T such that (24) holds for all $0 \leq t < T$ whenever (23) is satisfied. In addition, we obtain the following results by Proposition 3.3:

Lemma 3.12. *Let $\mathcal{L}_\alpha: dom(\mathcal{L}_\alpha) \subset H^1_\alpha(\mathbb{R})^2 \mapsto H^1_\alpha(\mathbb{R})^2$ be the operator defined in subsection 3.1. There exists $\nu > 0$ which satisfies*

$$sup\{Re\lambda: \lambda \in \sigma(\mathcal{L}_\alpha)\} < -\nu. \quad (25)$$

Moreover, there exists a constant $K > 0$ such that $\| e^{t\mathcal{L}_\alpha} \|_{H^1_\alpha(\mathbb{R})^2 \to H^1_\alpha(\mathbb{R})^2} \leq Ke^{-\nu t}$ for $t \geq 0$.

Proof. Recall equation (11):

$$L = \begin{pmatrix} 1 & 0 \\ 0 & \epsilon \end{pmatrix} \partial_{zz} + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} c\, \partial_z + \begin{pmatrix} 0 & e^{-\kappa} \\ 0 & -\kappa e^{-\kappa} \end{pmatrix}.$$

The operator $\mathcal{L}_\alpha$ generates an analytic semigroup provided $\epsilon > 0$ and a strongly continuous semigroup provided $\epsilon = 0$. As shown in [4], in both cases the differential operator $\mathcal{L}$ associated with the differential expression L in (11) enjoys the spectral mapping property, that is, the boundary of the spectrum of the semigroup operator $e^{t\mathcal{L}_\alpha}$ is controlled by the boundary of the spectrum of the semigroup generator $\mathcal{L}_\alpha$ for any $\epsilon \geq 0$. By Proposition 3.3 we can choose $\nu > 0$ such that $sup\{\text{Re}\,\lambda: \lambda \in \sigma(\mathcal{L}_\alpha)\} < -\nu$. Furthermore, by the above mentioned semigroup property, see, e.g. Proposition 4.3 in [4], there exists $K > 0$ such that $||e^{t\mathcal{L}_\alpha}||_{H^1_\alpha(\mathbb{R})^2 \to H^1_\alpha(\mathbb{R})^2} \leq Ke^{-\nu t}$. □

We are ready to start the stability analysis. We will first show that the solution of (10) is exponentially decaying in the $||\cdot||_\alpha$ norm.

Proposition 3.13. *Let $\nu > 0$ satisfies (25). Then there exist $\delta_1 > 0$ and $K_1 > 0$ such that for every $\delta \in (0, \delta_1)$ and every γ with $0 < \gamma < \delta$, the following holds for the mild solution $\mathbf{v} = \mathbf{v}(t)$ of (10) with the initial value $\mathbf{v}^0$: If $\mathbf{v}^0 \in \mathcal{E}^2$ satisfies (23) such that $\mathbf{v}(t)$ satisfies (24) for $0 \leq t < T(\delta, \gamma)$, then*

$$\big||\mathbf{v}(t)||_\alpha \leq K_1 e^{-\nu t}\big|\mathbf{v}^0||_\alpha, 0 \leq t < T(\delta, \gamma). \quad (26)$$

Proof. Since $\mathbf{v}(t)$ is a mild solution of (10) on $\mathcal{E}^2$, it satisfies the integral equation

$$\mathbf{v}(t) = e^{t\mathcal{L}_\mathcal{E}}\mathbf{v}^0 + \int_0^t e^{(t-s)\mathcal{L}_\mathcal{E}} H(\mathbf{v}(s))ds. \quad (27)$$

Since $\mathbf{v}^0 \in \mathcal{E}^2$ by assumption, it is clear that $H(\mathbf{v}(s))$ is in $H^1_\alpha(\mathbb{R})$ by Proposition 3.9, so we have

$$e^{t\mathcal{L}_\mathcal{E}}\mathbf{v}^0 = e^{t\mathcal{L}_\alpha}\mathbf{v}^0;$$
$$e^{(t-s)\mathcal{L}_\mathcal{E}} H(\mathbf{v}(s)) = e^{(t-s)\mathcal{L}_\alpha} H(\mathbf{v}(s)).$$

Next, we replace $\mathcal{L}_\mathcal{E}$ by $\mathcal{L}_\alpha$ in (27) and choose $\overline{\nu} > \nu > 0$ such that $sup\{\text{Re}\lambda: \lambda \in \sigma(\mathcal{L}_\alpha)\} < -\overline{\nu} < \nu < 0$ and define $k = \overline{\nu}/\nu > 1$.

By Lemma 3.12, there exists $K_1 > 0$ such that $||e^{t\mathcal{L}_\alpha}|| \leq K_1 e^{-\overline{\nu}t}$ for all $t \geq 0$. Pick any $\delta' > 0$, for $0 < \gamma < \delta'$, and notice that if $||\mathbf{v}^0||_\mathcal{E} < \gamma$, then $||\mathbf{v}(s)||_\mathcal{E} < \delta'$ for all $s \in (0, T(\delta', \gamma))$ by Lemma 3.11.

With the aid of Propostion 3.9. (1), there exists a constant $C_{\delta'} > 0$ depending on δ' such that for all $t \in [0, T(\delta', \gamma))$, using that $||\mathbf{v}(s)||_\mathcal{E} \leq \delta'$ when $s \in (0, T(\delta', \gamma))$, it follows that

$$\begin{aligned} &||\mathbf{v}(t)||_\alpha \\ &\leq K_1 e^{-\overline{\nu}t}||\mathbf{v}^0||_\alpha \\ &+ \int_0^t K_1 e^{-\overline{\nu}(t-s)} C_{\delta'}||\mathbf{v}(s)||_0||\mathbf{v}(s)||_\alpha ds. \end{aligned}$$

For each $\delta < \delta'$, and $0 < \gamma < \delta$, if $||\mathbf{v}^0||_\mathcal{E} < \gamma$, then $||\mathbf{v}(s)||_\mathcal{E} < \delta$ for all $s \in (0, T(\delta, \gamma))$ by Lemma 3.1. Then, for all $t \in [0, T(\delta', \gamma))$, we have

$$||\mathbf{v}(t)||_\alpha \leq K_1 e^{-\overline{\nu}t}||\mathbf{v}^0||_\alpha + K_1 C_{\delta'}\delta \int_0^t e^{-\overline{\nu}(t-s)}||\mathbf{v}(s)||_\alpha ds.$$

Applying Gronwall's inequality for $e^{\overline{\nu}t} \parallel \mathbf{v}(t) \parallel_\alpha$, we conclude that the inequality

$$e^{\overline{\nu}t}||\mathbf{v}(t)||_\alpha \leq K_1||\mathbf{v}^0||_\alpha + K_1 C_{\delta'}\delta \int_0^t e^{\overline{\nu}s}||\mathbf{v}(s)||_\alpha ds,$$

implies, by Gronwall's inequality, that

$$e^{\overline{\nu}t}\big|\mathbf{v}(t)||_\alpha \leq K_1\big|\mathbf{v}^0||_\alpha e^{K_1 C_{\delta'}\delta t},$$

so that

$$||\mathbf{v}(t)||_\alpha \leq K_1||\mathbf{v}^0||_\alpha e^{K_1 C_{\delta'}\delta t - \overline{\nu}t}.$$

By choosing $\delta_1 < min\{\delta', (k-1)\frac{\nu}{K_1 C_{\delta'}}\}$, we can conclude that (26) holds for all $\delta \in (0, \delta_1)$. □

We now show that the solution of (10) is bounded in the $||\cdot||_0$ norm, and the component $v_2(t)$ is exponentially decaying in the $||\cdot||_0$ norm.

Proposition 3.14. *Let $\rho > 0$ be chosen as in Lemma 3.4. (3), and δ_1 be given by Proposition 3.13. Assume that $\nu < \rho$, where ν satisfies (25). Then there exist constants $\delta_2 \in (0, \delta_1)$ and $C_1 > 0$ such that for every $\delta \in (0, \delta_2)$ and every γ with $0 < \gamma < \delta$, the following holds: If $0 \leq t < T(\delta, \gamma)$, and $\mathbf{v}^0 \in \mathcal{E}^2$ satisfies (23) such that $\mathbf{v}(t) \in \mathcal{E}^2$ satisfies (24), then the following estimates hold:*

$$||v_1(t)||_0 \leq C_1||\mathbf{v}^0||_\mathcal{E}, \quad (28)$$
$$||v_2(t)||_0 \leq C_1 e^{-\rho t}||\mathbf{v}^0||_\mathcal{E}. \quad (29)$$

Proof. We write (10) as a non-autonomous linear system on $H^1(\mathbb{R})^2$:

$$v_{1t} = L_1 v_1 + e^{-\kappa} v_2 + H_1(v_1(t), v_2(t)), \quad (30)$$
$$v_{2t} = L_2 v_2 + H_2(v_1(t), v_2(t)), \quad (31)$$

where L_1, L_2 are defined in (16) and (17), $\mathbf{v}(t) = (v_1, v_2)(t)$ is a fixed solution of (10), and

$$H_1(\mathbf{v}) = v_2(g(v_1 + 1/\kappa) - g(v_1)),$$
$$H_2(\mathbf{v}) = -\kappa v_2((g(v_1 + 1/\kappa) - g(v_1)).$$

Note that (v_1, v_2) is the solution of (30)-(31) with the value (v_1^0, v_2^0) at $t = 0$, that is $(v_1, v_2)(t) = (v_1, v_2)(t, v_1^0, v_2^0)$.

With the help of Proposition 3.8. (2), we can find a constant $C_{\delta_1} > 0$ such that

$$||H_1(v_1, v_2)||_0 \leq C_{\delta_1}||v_1||_0||v_2||_0, \quad (32)$$

and

$$||H_2(v_1, v_2)||_0 = \| -\kappa H_1(\mathbf{v}) \|_0 \leq C_{\delta_1}||v_1||_0||v_2||_0 \quad (33)$$

if $||\mathbf{v}||_0 \leq \delta_1$.

The solution of (31) in $H^1(\mathbb{R})$ can be written as

$$v_2(t) = e^{t\mathcal{L}_2}v_2^0 + \int_0^t e^{(t-s)\mathcal{L}_2} H_2(v_1(s), v_2(s))\, ds.$$

We then choose some $\overline{\rho} > \rho > 0$ and $k = \overline{\rho}/\rho > 1$ such that

$$sup\{\text{Re}\lambda: \lambda \in \sigma(\mathcal{L}_2)\} < -\overline{\rho} := -k\rho.$$

By Lemma 3.4. (3), there exists $K_2 > 0$ such that $||e^{t\mathcal{L}_2}||_{H^1(\mathbb{R})\to H^1(\mathbb{R})} \le K_2 e^{-\overline{\rho}t}$. For each $\delta \in (0, \delta_1)$ and $\gamma \in (0, \delta)$, if $||\mathbf{v}^0||_{\mathcal{E}} \le \gamma$ then

$$||v_1(s)||_0 \le ||v_1(s)||_{\mathcal{E}} \le ||\mathbf{v}(s)||_{\mathcal{E}} \le \delta.$$

By Lemma 3.11, we can use (33) to obtain the following estimate for $v_2(t)$:

$$\begin{aligned} &||v_2(t)||_0 \\ &\le K_2 e^{-\overline{\rho}t}||v_2^0||_0 \\ &+ \int_0^t K_2 e^{-\overline{\rho}(t-s)} C_{\delta_1} ||v_1(s)||_0 ||v_2(s)||_0\, ds \\ &\le K_2 e^{-\overline{\rho}t}||v_2^0||_0 \\ &+ \int_0^t K_2 e^{-\overline{\rho}(t-s)} C_{\delta_1} \delta ||v_2(s)||_0\, ds. \end{aligned}$$

We then compute

$$\begin{aligned} e^{\overline{\rho}t}||v_2(t)||_0 &\le K_2||v_2^0||_{\mathcal{E}} \\ &+ K_2 C_{\delta_1}\delta \int_0^t e^{\overline{\rho}s}||v_2(s)||_0\, ds \\ &\le K_2||\mathbf{v}^0||_{\mathcal{E}} \\ &+ K_2 C_{\delta_1}\delta \int_0^t e^{\overline{\rho}s}||v_2(s)||_0\, ds. \end{aligned}$$

Applying Gronwall's inequality to $e^{\overline{\rho}t}||v_2(t)||_0$, we infer that

$$||v_2(t)||_0 \le K_2||\mathbf{v}^0||_{\mathcal{E}} e^{K_2 C_{\delta_1}\delta t - \overline{\rho}t}.$$

Let $\delta_2 < min(\delta_1, \frac{(k-1)\rho}{K_2 C_{\delta_1}})$. Then for $\delta < \delta_2$ it follows that

$$\big||v_2(t)|\big|_0 \le K_2\big||\mathbf{v}^0|\big|_{\mathcal{E}} e^{-\rho t} \text{ for } t \in \big[0, T(\delta,\gamma)\big), \tag{34}$$

which proves (29).

We now give the proof of equation (28). The solution of (30) in $H^1(\mathbb{R})$ satisfies

$$\begin{aligned} v_1(t) = e^{t\mathcal{L}_1}v_1^0 + \int_0^t e^{(t-s)\mathcal{L}_1}(e^{-\kappa}v_2(s) \\ + H_1(v_1(s), v_2(s))\, ds. \end{aligned}$$

First, since $\mathcal{L}_1$ generates a bounded semigroup by Lemma 3.4. (1), there exists a constant $K_3 > 0$, such that $||e^{t\mathcal{L}_1}||_{H^1(\mathbb{R})\to H^1(\mathbb{R})} \le K_3$. Using equation (32) and the fact that $||e^{-\kappa}v_2(s)||_0 \le ||v_2(s)||_0$ for $\kappa > 0$, we infer that

$$\begin{aligned} ||v_1(t)||_0 &\le K_3||v_1^0||_0 \\ &+ \int_0^t (K_3 C_{\delta_1}||v_2(s)||_0||v_1(s)||_0 \\ &+ K_3||v_2(s)||_0)ds. \end{aligned}$$

Also, using the fact that $||v_1(s)||_0 \le ||\mathbf{v}(s)||_0 \le ||\mathbf{v}(s)||_{\mathcal{E}} < \delta < \delta_2$, we have, for a constant $C_{\delta_1,\delta_2} > 0$ independent of δ, that

$$\begin{aligned} ||v_1(t)||_0 &\le K_3||v_1^0||_{\mathcal{E}} + \int_0^t (K_3 C_{\delta_1,\delta_2}||v_2(s)||_0 \\ &+ K_3||v_2(s)||_0)ds \\ &\le K_3||v_1^0||_{\mathcal{E}} \\ &+ \int_0^t K_3 C_{\delta_1,\delta_2}||v_2(s)||_0\, ds. \end{aligned}$$

Then we use (34) to obtain

$$\begin{aligned} ||v_1(t)||_0 &\le K_3||\mathbf{v}^0||_{\mathcal{E}} \\ &+ \int_0^t K_2 K_3 C_{\delta_1,\delta_2} e^{-\rho s}||\mathbf{v}^0||_{\mathcal{E}}\, ds \\ &\le K_3||\mathbf{v}^0||_{\mathcal{E}} \\ &+ K_2 K_3 C_{\delta_1,\delta_2}||\mathbf{v}^0||_{\mathcal{E}} \int_0^t e^{-\rho s}\, ds \\ &\le C_2||\mathbf{v}^0||_{\mathcal{E}} \end{aligned}$$

for some $C_2 > 0$. In conclusion, there exists a constant $C_1 > 0$ such that for $\delta \in (0, \delta_2)$ and $\gamma \in (0, \delta)$, the inequalities (28) and (29) hold if $t \in [0, T(\delta,\gamma))$. □

We now complete the proof of the nonlinear stability of the end state $\mathbf{u}_-$.

Remark 3.15. *We claim that the end state $\mathbf{u}_-$ of (5) is stable in $||\cdot||_{\mathcal{E}}$. The proof of the stability of $\mathbf{u}_-$ is, in fact, contained in the next theorem and relies on on the following bootstrap argument based on Proposition 3.13 and Proposition 3.14. Indeed, these propositions yield the existence of constants $\delta_0 > 0$ and $C_{\delta_0} > 0$ such that for every $\delta \in (0, \delta_0)$ and every $\gamma \in (0, \delta)$, there exists $T(\delta,\gamma)$ such that for every $t \in [0, T(\delta,\gamma))$ the inequalities*

$$\| \mathbf{v}(t) \|_{\mathcal{E}} < \delta \text{ and } \| \mathbf{v}(t) \|_{\mathcal{E}} \le C_{\delta_0} \| \mathbf{v}^0 \|_{\mathcal{E}} \tag{35}$$

hold for the solution $\mathbf{v}(t)$ of (10) with initial value $\mathbf{v}^0 \in \mathcal{E}$ as long as $\| \mathbf{v}^0 \|_{\mathcal{E}} < \gamma$. Let us show that for each $\delta \in (0, \delta_0)$, there exists an η such that if $\| \mathbf{v}^0 \|_{\mathcal{E}} < \eta$ then $\| \mathbf{v}(t) \|_{\mathcal{E}} < \delta$ for all $t \ge 0$, that is, the end state $\mathbf{u}_-$ of (5) is stable in $\mathcal{E}$. Indeed, assuming $C_{\delta_0} > 1$ with no loss of generality, set $\eta = \frac{\delta}{2C_{\delta_0}}$ and assume $\| \mathbf{v}^0 \|_{\mathcal{E}} < \eta$. Then $\| \mathbf{v}(T(\delta,\gamma)) \|_{\mathcal{E}} < \delta/2$ by using (35), and thus the solution $\mathbf{v}$ with the initial value $\mathbf{v}(T(\delta,\gamma)))$ satisfies (35) again for $t \in [T(\delta,\gamma), 2T(\delta,\gamma))$, again by Propositions 3.13 and 3.14. So, these propositions can be applied for all $t \ge 0$, proving the stability. In addition, as long as these propositions are applicable, we obtain a more refined information about the behavior of the solution, such as its boundedness in $\|\cdot\|_0$ norm and the exponential decay in $\|\cdot\|_\alpha$ norm, see items (3)-(4) of the next theorem. We now proceed with a more formal exposition of the stability statement.

Given an initial value $\mathbf{v}^0 \in \mathcal{E}^2$, let $\mathbf{v}(t) = \mathbf{v}(t, \mathbf{v}^0)$ be the solution of (10) in $\mathcal{E}^2$ with $\mathbf{v}(0) = \mathbf{v}^0$, which we have shown to exist on $0 \le t < t_{max}(\mathbf{v}^0)$ by Lemma 3.10. We shall show that $\mathbf{v}(t) \in \mathcal{E}^2$ is defined and bounded in $\|\cdot\|_{\mathcal{E}}$ norm, and exponentially decaying in $\|\cdot\|_{\alpha}$ norm for all time $t > 0$. We note that the small constant in the next theorem can be chosen as $\delta_0 = \delta_2$ where δ_2 is chosen as in Proposition 3.14.

Theorem 3.16. *There exist constants $C > 0$ and $\nu > 0$ such that for each $0 < \delta < \delta_0$, we can find $\eta > 0$ such that if $||\mathbf{v}^0||_{\mathcal{E}} \le \eta$, then for all $t > 0$ the following holds:*
(1) $\mathbf{v}(t)$ is defined;
(2) $||\mathbf{v}(t)||_{\mathcal{E}} \le \delta$;
(3) $||\mathbf{v}(t)||_{\alpha} \le Ce^{-\nu t}||\mathbf{v}^0||_{\alpha}$;
(4) $||v_1(t)||_0 \le C||\mathbf{v}^0||_{\mathcal{E}}$;
(5) $||v_2(t)||_0 \le Ce^{-\nu t}||\mathbf{v}^0||_{\mathcal{E}}$.

Proof. Choose ν as in Lemma 3.12. Choose $\delta_0 = \delta_2$ as indicated in Proposition 3.14. Let C be a constant satisfying $C > max\{1, K_1, C_1\}$ with K_1 and C_1 given as in Propositions 3.13 and 3.14. Let $0 < \gamma < \delta < \delta_0$ and set $\eta = C^{-1}\gamma$. Assume $\mathbf{v}^0 \in \mathcal{E}$ satisfies $\| \mathbf{v}^0 \|_{\mathcal{E}} \le \eta$. Since $\| \mathbf{v}^0 \|_{\mathcal{E}} < \eta < \delta$, the solution $\mathbf{v}(t)$ exists and satisfies statements (2)-(5) in the theorem for $t \in [0, T(\delta, \eta))$ by Propositions 3.13 and 3.14. We claim that $T(\delta, \eta) = \infty$, so that the proof is finished as soon as the claim is justified. To do this, for any $T \in (0, T(\delta, \eta))$ we consider the solution with the initial data $\mathbf{v}(T)$. Note that staments (3)-(5) for $t = T$ yield $\| \mathbf{v}(T) \|_{\mathcal{E}} \le C \| \mathbf{v}^0 \|_{\mathcal{E}} \le C\eta \le \gamma$ and thus Lemma 3.11 applies and gives $\| \mathbf{v}(T + t) \|_{\mathcal{E}} \le \delta$ for $t \in (0, T(\delta, \gamma))$. Therefore, we proved that if $\| \mathbf{v}^0 \|_{\mathcal{E}} < \eta$ then $||\mathbf{v}(t)||_{\mathcal{E}} \le \delta$ for all $t \in [0, T(\delta, \gamma) + T)$. This shows that $T(\delta, \eta) \ge T(\delta, \gamma) + T$ and therefore implies $T(\delta, \eta) \ge T(\delta, \gamma) + T(\delta, \eta)$ and thus $T(\delta, \eta) = \infty$ as claimed.

□

4 Conclusion

We now generalize the above results to a more general system with the general nonlinearity $f(\cdot)$ and the coefficient matrix D given by

$$\mathbf{u}_t(t, \mathbf{x}) = D\Delta_{\mathbf{x}}\mathbf{u}(t, \mathbf{x}) + f(\mathbf{u}(t, \mathbf{x})), \quad (36)$$

where $\mathbf{u} \in \mathbb{R}^{\mathrm{n}}, \mathbf{x} \in \mathbb{R}^{\mathrm{d}}, t \ge 0, D = diag(d_1, \dots, d_n)$ with all $d_i \ge 0$, and the function $f: \mathbb{R}^n \to \mathbb{R}^n$ is smooth, see Hypothesis 4.1 below. We will present here the stability result of an $\mathbf{x}$-independent stationary solution $\mathbf{u}_- = 0$ to the system (36) and its perturbation depending only on $z = \mathbf{x} \cdot \mathbf{e} - ct$ from summarizing the stability analysis of the model system (5).

Hypothesis 4.1.
(a) *In appropriate variables $\mathbf{u} = (\mathbf{u}_1, \mathbf{u}_2), \mathbf{u}_1 \in \mathbb{R}^{n_1}, \mathbf{u}_2 \in \mathbb{R}^{n_2}$, $n_1 + n_2 = n$, we assume that for some constant $n_1 \times n_1$ matrix A_1,*
$$f(\mathbf{u}_1, 0) = (A_1\mathbf{u}_1, 0)^T.$$
Moreover,
$$D = \begin{pmatrix} D_1 & 0 \\ 0 & D_2 \end{pmatrix}, f(\mathbf{u}) = \begin{pmatrix} f_1(\mathbf{u}_1, \mathbf{u}_2) \\ f_2(\mathbf{u}_1, \mathbf{u}_2) \end{pmatrix}$$
where each D_i is a nonnegative diagonal matrix of size $n_i \times n_i$, and $f_i: \mathbb{R}^{n_1} \times \mathbb{R}^{n_2} \to \mathbb{R}^{n_i}$ for $i = 1,2$.
(b) *The function f is C^3 from $\mathbb{R}^n$ to $\mathbb{R}^n$.*
(c) *For the linear operator $\mathcal{L}_\alpha$ associated with the differential expression*
$$L = D\partial_{zz} + c\partial_z + \partial_{\mathbf{u}} f(0,0),$$
there exists $\alpha > 0$ such that $sup\{Re\, \lambda: \lambda \in \sigma(\mathcal{L}_\alpha)\} < 0$ on $H^1_\alpha(\mathbb{R})^n$.
(d) *The operator $\mathcal{L}_1$ associated with the differential expression*
$$L_1 = D_1\partial_{zz} + c\partial_z + A_1$$
generates a bounded semigroup on $L^2(\mathbb{R})^{n_1}$ and $H^1(\mathbb{R})^{n_1}$.
(e) *The operator $\mathcal{L}_2$ associated with the differntial expressin*
$$L_2 = D_2\partial_{zz} + c\partial_z + \partial_{\mathbf{u}_2} f_2(0,0)$$
satisfies $sup\{Re\, \lambda: \lambda \in \sigma(\mathcal{L}_2)\} < 0$ on $L^2(\mathbb{R})^{n_2}$ and $H^1(\mathbb{R})^{n_2}$.

By the discussion in Section 2, replacing the spatial variable $\mathbf{x} = (x_1, \dots, x_d)$ by the moving variable $z = \mathbf{x} \cdot \mathbf{e} - ct \in \mathbb{R}$ in (36), we obtain

$$\mathbf{u}_t(t, z) = D\mathbf{u}_{zz}(t, z) + c\mathbf{u}_z(t, z) + f(\mathbf{u}(t, z)), \quad (37)$$

We will now rewrite the equation for the perturbation $\mathbf{v}(t, z)$ of the stationary solution $\mathbf{u}_-$ in the form amenable for the subsequence analysis. We seek a solution to (37) of the form $\mathbf{u}(t, z) = 0 + \mathbf{v}(t, z)$, with this notation, $\mathbf{v}(t, z)$ satisfies

$$\mathbf{v}_t = D\mathbf{v}_{zz} + c\mathbf{v}_z + \partial_{\mathbf{u}} f(0)\mathbf{v} + f(\mathbf{v}) - f(0) - \partial_{\mathbf{u}} f(0)\mathbf{v}. \quad (38)$$

We can show that for the system (36) satisfying Hypothesis 4.1, if the initial values of the perturbation of the stationary solution $\mathbf{u}_-$ are sufficiently small in both the weighted and unweighted norms, the perturbation will converge exponentially in the global time domain in the weighted norm and remain bounded in the unweighted norm. Note that the hypotheses we have given are sufficient to cover the conditions we need in the proof, so this proof will be very similar to the one in Section 3, and we will not repeat it here.

In summary, in this paper we have discussed the stability of the stationary solution of a class of reaction-diffusion equations in multidimensional space and have summarized the characteristics of this class of equations. Although the approach of this discussion is essentially similar to the one in [5] for equations in one-dimensional space, it can be constituted together with the discussion in [7] for the stability of the stationary solution of this class of reaction-diffusion equations in multidimensional space with respect to two types of perturbations. It is worth noting that there are still many unsolved problems in this type of nonlinear stability analysis, such as the stability analysis of traveling wave solutions $\mathbf{w}(\mathbf{x})$ for this class of equations in multidimensional space, which we mentioned in Section 2, is still a difficult problem, and this is a direction that may need to be covered in future studies.

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Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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