Finite-time Stochastic Stability and Stabilization for Uncertain Discrete-time Stochastic Systems with Time-varying Delay

XINYUE TANG, YALI DONG*, MENG LIU,

School of Mathematical Sciences
Tiangong University
Tianjin 300387
CHINA

Abstract: This paper deals with the problems of finite-time stochastic stability and stabilization for discrete-time stochastic systems with parametric uncertainties and time-varying delay. Using the Lyapunov-Krasovskii functional method, some sufficient conditions of finite-time stochastic stability for a class of discrete-time stochastic uncertain system^{S} are established in term of matrix inequalities. Then, a new criterion is proposed to ensure the closed-loop system is finite-time stochastically stable. The controller gain is designed. Finally, two numerical examples are given to illustrate the effectiveness of the proposed results.

Key-Words: Finite-time stochastic stability; finite -time stochastic stabilization; discrete-time stochastic systems; time-varying delay; parametric uncertainty.

Received: January 5, 2021. Revised: August 14. Accepted: September 13, 2021. Published: September 22, 2021.

1. Introduction

Classical concepts of stability, such as Lyapunov stability or BIBO stability, mainly deal with systems running in infinite time intervals. The value of the boundary is generally not specified. But in many practical applications, the state of the system is expected to not exceed a certain domain in a finite time interval. So, along with the classical Lyapunov stability, one is also concerned about the finite-time transient performance. A system is finite-time stable if, once we fix a time interval and give a bound on the initial condition, the system state does not exceed a certain domain during this time interval. Recently, finite-time stability has gradually become a hot topic and has been applied to many systems, such as continuous systems [1-3], discrete systems [4-6], stochastic systems [7, 8], and switched systems [9-11].

The phenomenon of time-delay is very common in practical engineering systems, such as chemical systems, biological systems, mechanical systems, and networked control systems. The existence of time-delay is the root cause of the instability and poor performance of the control system. In most cases, the time-delay is not constant but time-varying. Many researchers were engaged in the study of time-delay systems (see e.g. [4, 9, 12-14] and references therein). Stojanovic [12] dealt with the problem of robust finite-time stability for discrete time delay systems with nonlinear perturbations. In [13], finite-time stability of linear discrete-time systems with time-varying delay was considered. In [15], Arunkumar et al. studied robust stability criteria for discrete-time switched neural networks. Zuo et al. [16] considered the finite-time stochastic stability and stabilization for linear discrete-time Markovian jump systems. In [17], finite-time stability and stabilization results for switched impulsive dynamical systems on time scales were proposed. Li et al. [18] dealt with the problem of finite-time stability for time-varying time-delay systems. Moradi [19] looked at the problem of finite-time stability for time-varying time-delay systems. To the best of our knowledge, the problems of finite-time stability and finite-time stabilization for discrete-time stochastic systems are important and have not been fully discussed, which leads to the main purpose of our research

In this paper, we consider finite-time stochastic stability and finite-time stochastic stabilization for discrete-time uncertain stochastic systems with time-varying delay. First, we develop sufficient conditions of finite-time stochastic stability for the open-loop discrete-time stochastic systems. Then we present sufficient condition such that the resulting closed-loop system is finite-time stochastically stable for all admissible uncertainties. We design a state feedback controller. We provide two numerical examples to demonstrate the validity of the proposed approach.

The rest of this paper is organized as follows. Some preliminaries and the problem statement are described in Section 2. In Section 3, the sufficient conditions of the finite-time stochastic stability and stabilization for uncertain discrete time-varying delay stochastic systems are established. Two numerical examples are presented in Section 4. Some conclusions are drawn in Section 5.

Notations. The superscript "T" denotes the transpose; $M<0(M>0)$ denotes the matrix M is a negative definite (positive definite) symmetric matrix; $E\{\cdot\}$ stands for mathematical expectation operator with respect to the given probability measure P; $\lambda_{\max}(.)$ and $\lambda_{\min}(.)$ denotes the maximum eigenvalue and minimum eigenvalue of a matrix respectively; $\mathbb{N}$ denotes the non-negative integer set. The asterisk $*$ in a matrix is used to denote term that is induced by symmetry.

2. Problem Formulation

Consider the following uncertain discrete-time stochastic system with time-varying delay

$$\begin{cases} x(k+1)=A_1(k)x(k)+A_d(k)x(k-d(k)), \\ \quad +B(k)u(k)+C_1(k)x(k)\omega(k) \\ \quad +C_2(k)x(k-d(k))\omega(k), \\ x(\theta)=\varphi(\theta), \forall\theta\in[-d_M,-d_M+1,\cdots,0], \end{cases} \tag{1}$$

where $x(k)\in R^n$ is the n-dimensional state vector; $u(k)\in R^p$ is the control input; $d(k)$ is the positive integer representing the time-varying with

$$d_m \le d(k) \le d_M .$$

$\{\omega(k)\}_{k\in\mathbb{N}}$ is a sequence of one-dimensional independent white noise processes defined on the complete filtered probability space with

$$E\{\omega(k)\}=0, \quad E\{\omega^2(k)\}=1.$$

$\varphi(k)$ is the initial condition. The matrices $A_1(k)$, $A_d(k)$, $B(k)$, $C_1(k)$ and $C_2(k)$ are time-varying matrices, which are assumed to be of the form:

$$\begin{aligned} &A_1(k)=A_1+\Delta A_1(k),\ B(k)=B+\Delta B(k), \\ &A_d(k)=A_d+\Delta A_d(k),\ C_1(k)=C_1+\Delta C_1(k), \\ &C_2(k)=C_2+\Delta C_2(k). \end{aligned} \tag{2}$$

A_1, A_d, B, C_1, C_2 are known real constant matrices. $\Delta A_1(k)$, $\Delta A_d(k)$, , $\Delta B(k)$, $\Delta C_1(k)$ and $\Delta C_2(k)$ are unknown matrices representing time-varying parameter uncertainties and are assumed to be of the following form:

$$\begin{aligned} &[\Delta A_1(k) \quad \Delta A_d(k) \quad \Delta C_1(k) \quad \Delta C_2(k)] \\ &\quad = MF(k)[N_1 \quad N_2 \quad N_3 \quad N_4], \\ &\Delta B(k)=MF(k)N_b, \end{aligned} \tag{3}$$

where M and $N_b, N_i(i=1,2,3,4)$ are known real constant matrices and $F(k)$ is the unknown time-varying matrix-valued function subject to the following condition :

$$F^T(k)F(k)\le I, \quad \forall k\in\mathbb{N}. \tag{4}$$

Before presenting the main results, some useful definition and lemmas are given.

Definition 1. Given positive constants c_1, c_2 and N with $c_1<c_2$, system (1) with $u(k)=0$ is said to be finite-time stochastically stable with respect to (c_1,c_2,N), if

$$\begin{aligned} &\sup_{k\in[-d_M,-d_M+1,\cdots 0]} \varphi^T(k)\varphi(k)<c_1 \\ &\quad \Rightarrow E\{x^T(k)x(k)\}<c_2,\ \forall k\in[1,2,\cdots N]. \end{aligned} \tag{5}$$

Lemma 1. [20] Let D, S and F be real matrices of appropriate dimension with F satisfying $F^TF\le I$. Then, for any scalar $\varepsilon>0$,

$$DFS+(DFS)^T\le\varepsilon^{-1}DD^T+\varepsilon S^TS. \tag{6}$$

Lemma 2. [15] Given constant matrices Ω_1, Ω_2, Ω_3, where $\Omega_1=\Omega_1^T>0$ and $\Omega_2^T=\Omega_2>0$, then

$$\Omega_1+\Omega_3^T\Omega_2^{-1}\Omega_3<0,$$

if and only if

$$\begin{bmatrix} \Omega_1 & \Omega_3^T \\ \Omega_3 & -\Omega_2 \end{bmatrix}<0. \tag{7}$$

The aim of this paper is to develop some sufficient conditions to ensure system (1) is finite-time stochastic stabilization.

3. Main Results

In this section, we will first give the finite-time stochastic stability condition for discrete-time stochastic system. Further, we design the stabilizing controllers for the system (1).

3.1. Finite-time stochastic stability

Theorem 1. System (1) with $u(k)=0$ is finite-time stochastically stable with respect to (c_1,c_2,N), if there exist symmetric positive definite matrices P, Q, and scalars $\gamma\ge1$, $\varepsilon_1>0$, $\varepsilon_2>0$, such that the following inequalities hold

$$\gamma^N[\lambda_{\max}(Q)c_1(d_M-d_m+1)(d_M-d_m)/2 + \lambda_{\max}(P)c_1+\lambda_{\max}(Q)d_Mc_1] < c_2\lambda_{\min}(P), \quad (8)$$

$$\begin{bmatrix} \Lambda & 0 & A_1^TP & C_1^TP & 0 & \varepsilon_1N_1^T & 0 & \varepsilon_2N_3^T \\ * & -Q & A_d^TP & C_2^TP & 0 & \varepsilon_1N_2^T & 0 & \varepsilon_2N_4^T \\ * & * & -P & 0 & PM & 0 & 0 & 0 \\ * & * & * & -P & 0 & 0 & PM & 0 \\ * & * & * & * & -\varepsilon_1I & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_1I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_2I & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2I \end{bmatrix} < 0, \quad (9)$$

where $\Lambda=(d_M-d_m+1)Q-\gamma P.$

Proof. Choose a Lyapunov-Krasovskii functional for system (1) as:

$$V(k)=V_1(k)+V_2(k)+V_3(k), \quad (10)$$

where

$$V_1(k)=x^T(k)Px(k),$$
$$V_2(k)=\sum_{i=k-d(k)}^{k-1}x^T(i)Qx(i),$$
$$V_3(k)=\sum_{j=k-d_M}^{k-d_m}\sum_{i=j}^{k-1}x^T(i)Qx(i).$$

Let $\Delta V(k+1)=V(k+1)-V(k)$, then we have

$$\begin{aligned} E\{\Delta V_1(k)\} &= E\left\{x^T(k+1)Px(k+1)-x^T(k)Px(k)\right\} \\ &= E\left\{x^T(k)A_1^T(k)PA_1(k)x(k)\right. \\ &+x^T(k)A_1^T(k)PA_d(k)x(k-d(k)) \\ &+x^T(k-d(k))A_d^T(k)PA_1(k)x(k) \\ &+x^T(k-d(k))A_d^T(k)PA_d(k)x(k-d(k)) \\ &+x^T(k)C_1^T(k)PC_1(k)x(k) \\ &+x^T(k)C_1^T(k)PC_2(k)x(k-d(k)) \\ &+x^T(k-d(k))C_2^T(k)PC_1(k)x(k) \\ &\left.+x^T(k-d(k))C_2^T(k)PC_2(k)x(k-d(k))\right\}, \end{aligned} \quad (11)$$

$$\begin{aligned} E\{\Delta V_2(k)\} &= E\{\sum_{i=k+1-d(k+1)}^{k}x^T(i)Qx(i) \\ &-\sum_{i=k-d(k)}^{k-1}x^T(i)Qx(i)\} \\ &= E\{x^T(k)Qx(k) \\ &-x(k-d(k))^TQx(k-d(k)) \\ &+\sum_{i=k-d_m+1}^{k-1}x^T(i)Qx(i)+\sum_{i=k-d(k)+1}^{k-d_m}x^T(i)Qx(i) \\ &-\sum_{i=k-d(k)+1}^{k-1}x^T(i)Qx(i)\} \\ &\le E\{x^T(k)Qx(k)-x^T(k-d(k))Qx(k-d(k)) \\ &+\sum_{i=k-d_M+1}^{k-d_m}x^T(i)Qx(i)\}, \end{aligned} \quad (12)$$

$$\begin{aligned} E\{\Delta V_3(k)\} &= E\{\sum_{j=k-d_M+1}^{k-d_m+1}\sum_{i=j}^{k}x^T(i)Qx(i) \\ &-\sum_{j=k-d_M+1}^{k-d_m}\sum_{i=j}^{k-1}x^T(i)Qx(i)\} \\ &= E\{\sum_{k-d_M+1}^{k-d_m}\sum_{i=j+1}^{k}x^T(i)Qx(i)-\sum_{j=k-d_M+1}^{k-d_m}\sum_{i=j}^{k-1}x^T(i)Qx(i)\} \\ &= E\{\sum_{j=k-d_M+1}^{k-d_m}(x^T(k)Qx(k)-x^T(j)Qx(j))\} \\ &= E\{(d_M-d_m)x^T(k)Qx(k)-\sum_{j=k-d_M+1}^{k-d_m}x^T(j)Qx(j)\} \\ &= E\{(d_M-d_m)x^T(k)Qx(k)-\sum_{j=k-d_M+1}^{k-d_m}x^T(j)Qx(j)\} \end{aligned} \quad (13)$$

From (11)-(13), it follows that

$$E\{\Delta V(k)\}\le E\{\xi(k)^T\Omega\xi(k)\}, \quad (14)$$

Where

$$\xi(k)=[x^T(k) \quad x^T(k-d(k))]^T,$$
$$\Omega=\begin{bmatrix}\Omega_{11} & \Omega_{12}\\ * & \Omega_{22}\end{bmatrix},$$
$$\Omega_{11}=(d_M-d_m+1)Q-P+A_1^T(k)PA_1(k)+C_1^T(k)PC_1(k),$$
$$\Omega_{12}=A_1^T(k)PA_d(k)+C_1^T(k)PC_2(k),$$
$$\Omega_{22}=-Q+A_d^T(k)PA_d(k)+C_2^T(k)PC_2(k).$$

Let

$$\Gamma=\Omega-\begin{bmatrix}(\gamma-1)P & 0\\ 0 & 0\end{bmatrix} =\begin{bmatrix}\Gamma_{11} & \Omega_{12}\\ * & \Omega_{22}\end{bmatrix}, \quad (15)$$

where

$$\Gamma_{11}=A_1^T(k)PA_1(k)+(d_M-d_m+1)Q-\gamma P+C_1^T(k)PC_1(k).$$

Note that Γ can be rewritten as:

$$\Gamma=\Gamma_1+\Gamma_2+\Gamma_3, \quad (16)$$

where

$$\Gamma_1 = \begin{bmatrix} (d_M - d_m + 1)Q - \gamma P & 0 \\ 0 & -Q \end{bmatrix},$$

$$\Gamma_2 = \begin{bmatrix} A_1^T(k)PA_1(k) & A_1^T(k)PA_d(k) \\ A_d^T(k)PA_1(k) & A_d^T(k)PA_d(k) \end{bmatrix},$$

$$\Gamma_3 = \begin{bmatrix} C_1^T(k)PC_1(k) & C_1^T(k)PC_2(k) \\ C_2^T(k)PC_1(k) & C_2^T(k)PC_2(k) \end{bmatrix}.$$

Note that Γ_2, Γ_3 can be rewritten as:

$$\begin{aligned} \Gamma_2 &= \begin{bmatrix} A_1^T(k) \\ A_d^T(k) \end{bmatrix} P \begin{bmatrix} A_1(k) & A_d(k) \end{bmatrix} \\ &= \Psi_1^T P \Psi_1, \end{aligned} \tag{17}$$

$$\begin{aligned} \Gamma_3 &= \begin{bmatrix} C_1^T(k) \\ C_2^T(k) \end{bmatrix} P \begin{bmatrix} C_1(k) & C_2(k) \end{bmatrix} \\ &= \Psi_2^T P \Psi_2. \end{aligned} \tag{18}$$

By (16), (17), (18) and Schur complement, $\Gamma < 0$ is equivalent to

$$\Phi = \begin{bmatrix} \Lambda & 0 & A_1^T(k)P & C_1^T(k)P \\ * & -Q & A_d^T(k)P & C_2^T(k)P \\ * & * & -P & 0 \\ * & * & * & -P \end{bmatrix} < 0, \tag{19}$$

where $\Lambda = (d_M - d_m + 1)Q - \gamma P.$

Φ can be written as:

$$\Phi = \tilde{\Phi} + \Delta\Phi,$$

where

$$\tilde{\Phi} = \begin{bmatrix} (d_M - d_m + 1)Q - \gamma P & 0 & A_1^T P & C_1^T P \\ * & -Q & A_d^T P & C_2^T P \\ * & * & -P & 0 \\ * & * & * & -P \end{bmatrix},$$

$$\begin{aligned} \Delta\Phi &= \begin{bmatrix} 0 & 0 & \Delta A_1^T P & \Delta C_1^T P \\ * & 0 & \Delta A_d^T P & \Delta C_2^T P \\ * & * & 0 & 0 \\ * & * & * & 0 \end{bmatrix} \\ &= D_1 F(k) S_1 + (D_1 F(k) S_1)^T + D_2 F(k) S_2 \\ &\quad + (D_2 F(k) S_2)^T, \end{aligned}$$

$$D_1 = \begin{bmatrix} 0 & 0 & M^T P & 0 \end{bmatrix}^T,$$

$$D_2 = \begin{bmatrix} 0 & 0 & 0 & M^T P \end{bmatrix}^T,$$

$$S_1 = \begin{bmatrix} N_1 & N_2 & 0 & 0 \end{bmatrix},$$

$$S_2 = \begin{bmatrix} N_3 & N_4 & 0 & 0 \end{bmatrix}.$$

From Lemma 1, we have

$$\Delta\Phi \le \varepsilon_1^{-1} D_1 D_1^T + \varepsilon_1 S_1^T S_1 + \varepsilon_2^{-1} D_2 D_2^T + \varepsilon_2 S_2^T S_2. \tag{21}$$

From Lemma 1 and (9), it can be seen that $\Phi < 0$, which implies that

$$\Gamma < 0. \tag{22}$$

Then

$$\begin{aligned} E\{\Delta V(x(k))\} &\le E\{\xi^T(k)\Omega\xi(k)\} \\ &= E\{\xi^T(k)(\Gamma + \begin{bmatrix} (\gamma-1)P & 0 \\ 0 & 0 \end{bmatrix})\xi(k)\} \\ &= E\{\xi^T(k)\Gamma\xi(k)\} + E\{\xi^T(k)\begin{bmatrix} (\gamma-1)P & 0 \\ 0 & 0 \end{bmatrix}\xi(k)\} \\ &< E\{\xi^T(k)\begin{bmatrix} (\gamma-1)P & 0 \\ 0 & 0 \end{bmatrix}\xi(k)\} \\ &= E\{(\gamma-1)x^T(k)Px(k)\} \\ &< E\{(\gamma-1)V(x(k))\}, \end{aligned}$$

which implies that

$$E\{V(k+1)\} < \gamma E\{V(k)\}. \tag{23}$$

From (23), we have

$$\begin{aligned} E\{V(k)\} &< \gamma E\{V(k-1)\} \\ &< \cdots \\ &< \gamma^k V(0) \\ &< \gamma^N V(0). \end{aligned} \tag{24}$$

From (10), we have

$$\begin{aligned} V(0) &= x^T(0)Px(0) + \sum_{i=-d(0)}^{-1} x^T(i)Qx(i) \\ &\quad + \sum_{j=-d_M}^{-d_m}\sum_{i=j}^{-1} x^T(i)Qx(i) \\ &< \lambda_{\max}(P)x^T(0)x(0) + \lambda_{\max}(Q)\sum_{i=-d_M}^{-1} x^T(i)x(i) \\ &\quad + \lambda_{\max}(Q)\sum_{j=-d_M}^{j=-d_m}\sum_{i=j}^{-1} x^T(i)x(i) \\ &< \lambda_{\max}(P)c_1 + \lambda_{\max}(Q)\sum_{i=-d_M}^{-1} c_1 + \lambda_{\max}(Q)\sum_{j=-d_M}^{-d_m}\sum_{i=j}^{-1} c_1 \\ &= \lambda_{\max}(P)c_1 + \lambda_{\max}(Q)d_M c_1 \\ &\quad + \lambda_{\max}(Q)c_1(d_M - d_m + 1)(d_M - d_m)/2 \end{aligned} \tag{25}$$

$$\begin{aligned} E\{V(k)\} &> E\{V_1(k)\} = E\{x^T(k)Px(k)\} \\ &> E\{\lambda_{\min}(P)x^T(k)x(k)\} \\ &= \lambda_{\min}(P)E\{x^T(k)x(k)\}. \end{aligned} \tag{26}$$

So, from (24)-(26), one get

$$\begin{aligned} \lambda_{\min}(P)E\{x^T(k)x(k)\} &< \gamma^N[\lambda_{\max}(P)c_1 + \lambda_{\max}(Q)d_M c_1 \\ &\quad + \lambda_{\max}(Q)c_1(d_M - d_m + 1)(d_M - d_m)/2] \end{aligned} \tag{27}$$

Using (8), we get that

$$E\{x^T(k)x(k)\} < c_2,$$

According to Definition 1, system (1) with $u(k)=0$ is finite-time stochastically stable. This completes the proof.

Remark 1. For this constant delay case, $d_m = d_M = d$, the following corollary can be obtained.

Corollary 1. System given by Eq. (1) with $d(k)=d$ and $u(k)=0$ is finite-time stochastically stable with respect to (c_1, c_2, N), if there exist symmetric positive definite matrices P, Q and scalar $\gamma \geq 1, \varepsilon_1 > 0, \varepsilon_2 > 0$, such that the following inequalities hold:

$$\gamma^N c_1[\lambda_{\max}(P) + \lambda_{\max}(Q)d] < c_2\lambda_{\min}(P), \quad (28)$$

$$\begin{bmatrix} Q-\gamma P & 0 & A_1^T P & C_1^T P & 0 & \varepsilon_1 N_1^T & 0 & \varepsilon_2 N_3^T \\ * & -Q & A_d^T P & C_2^T P & 0 & \varepsilon_1 N_2^T & 0 & \varepsilon_2 N_4^T \\ * & * & -P & 0 & PM & 0 & 0 & 0 \\ * & * & * & -P & 0 & 0 & PM & 0 \\ * & * & * & * & -\varepsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_1 I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0. \quad (29)$$

Proof. Let us select the following Lyapunov-Krasovskii function

$$V(k) = V_1(k) + V_2(k), \quad (30)$$

where

$$V_1(k) = x^T(k)Px(k),$$

$$V_2(k) = \sum_{i=k-d}^{k-1} x^T(i)Qx(i).$$

We have

$$\begin{aligned} E\{\Delta V_1(k)\} &= E\left\{x^T(k+1)Px(k+1) - x^T(k)Px(k)\right\} \\ &= E\left\{x^T(k)A_1^T(k)PA_1(k)x(k)\right. \\ &\quad + x^T(k)A_1^T(k)PA_d(k)x(k-d) \\ &\quad + x^T(k-d)A_d^T(k)PA_1(k)x(k) \\ &\quad + x^T(k-d)A_d^T(k)PA_d(k)x(k-d) \\ &\quad + x^T(k)C_1^T(k)PC_1(k)x(k) \\ &\quad + x^T(k)C_1^T(k)PC_2(k)x(k-d) \\ &\quad + x^T(k-d)C_2^T(k)PC_1(k)x(k) \\ &\quad \left. + x^T(k-d)C_2^T(k)PC_2(k)x(k-d)\right\}, \end{aligned}$$

$$\begin{aligned} E\{\Delta V_2(k)\} &= E\{\sum_{i=k+1-d}^{k} x^T(i)Qx(i) - \sum_{i=k-d}^{k-1} x^T(i)Qx(i)\} \\ &= E\left\{x^T(k)Qx(k) - x^T(k-d)Qx(k-d)\right\}. \end{aligned}$$

Then, it follows that

$$E\{\Delta V(k)\} = E\{\tilde{\xi}(k)^T \tilde{\Omega}\tilde{\xi}(k)\}, \quad (31)$$

where

$$\tilde{\xi}(k) = [x^T(k) \quad x^T(k-d)]^T,$$

$$\tilde{\Omega} = \begin{bmatrix} \tilde{\Omega}_{11} & \Omega_{12} \\ * & \tilde{\Omega}_{22} \end{bmatrix},$$

$$\tilde{\Omega}_{11} = Q - P + A_1^T(k)PA_1(k) + C_1^T(k)PC_1(k),$$

$$\Omega_{12} = A_1^T(k)PA_d(k) + C_1^T(k)PC_2(k),$$

$$\tilde{\Omega}_{22} = -Q + A_d^T(k)PA_d(k) + C_2^T(k)PC_2(k).$$

Just like the steps in Theorem 1, we have

$$E\{V(k)\} < \gamma^N V(0).$$

From (30), it follows that

$$\begin{aligned} V(0) &= x^T(0)Px(0) + \sum_{i=-d}^{-1} x^T(i)Qx(i) \\ &< \lambda_{\max}(P)x^T(0)x(0) + \lambda_{\max}(Q)\sum_{i=-d}^{-1} x^T(i)x(i) \\ &< \lambda_{\max}(P)c_1 + \lambda_{\max}(Q)\sum_{i=-d}^{-1} c_1 \\ &= \lambda_{\max}(P)c_1 + \lambda_{\max}(Q)dc_1, \end{aligned}$$

and

$$E\{V(k)\} > \lambda_{\min}(P)E\{x^T(k)x(k)\}.$$

So, we get

$$\lambda_{\min}(P)E\{x^T(k)x(k)\} < \gamma^N[\lambda_{\max}(P)c_1 + \lambda_{\max}(Q)dc_1].$$

Using (28), it follows that

$$E\{x^T(k)x(k)\} < c_2.$$

By Definition 1, one know that system (1) is with $d(k)=d$ and $u(k)=0$ is finite-time stochastically stable. This completes the proof.

3.2. Finite-time stochastic stabilization

Consider a state feedback control

$$u(k) = Kx(k). \quad (32)$$

From (1) and (32), we have the following closed-loop system

$$\begin{cases} x(k+1) = (A_1(k) + B(k)K)x(k) + A_d(k)x(k-d(k)) \\ +C_1(k)x(k)\omega(k) + C_2(k)x(k-d(k))\omega(k) \\ x = \varphi(k), \forall k \in [-d_M, -d_M+1, \cdots 0]. \end{cases} \quad (33)$$

Definition 2. Given positive constants c_1, c_2 and N with $c_1 < c_2$, system (1) is said to be finite-time stochastically stabilizable with respect to (c_1, c_2, N), if there exists a state feedback controller $u(k) = Kx(k)$, such that the resulting closed-loop system (33) is finite-time stochastically stable with respect to (c_1, c_2, N).

Theorem 2. System (1) is finite-time stochastically stabilizable with respect to (c_1, c_2, N) via a state feedback $u(k) = Kx(k)$, if there exists symmetric matrices $X > 0$, $\tilde{Q} > 0$, any matrix Y, scalars $\gamma \geq 1$, $\varepsilon_1 > 0, \varepsilon_2 > 0,$, such that the following inequalities hold:

$$\begin{bmatrix} (d_M - d_m + 1)\tilde{Q} - \gamma X^T & 0 & (A_1 X + BY)^T & XC_1^T & 0 & \varepsilon_1 (N_1 X + N_b Y)^T & 0 & \varepsilon_2 X N_3^T \\ * & -\tilde{Q} & XA_d^T & XC_2^T & 0 & \varepsilon_1 X N_2^T & 0 & \varepsilon_2 X N_4^T \\ * & * & -X & 0 & M & 0 & 0 & 0 \\ * & * & * & -X & 0 & 0 & M & 0 \\ * & * & * & * & -\varepsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_1 I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0, \tag{34}$$

$$\begin{aligned} &\gamma^N [\lambda_{\max}(Q) c_1 (d_M - d_m + 1)(d_M - d_m)/2 \\ &\quad + \lambda_{\max}(P) c_1 + \lambda_{\max}(Q) d_M c_1] < c_2 \lambda_{\min}(P), \end{aligned} \tag{35}$$

where

$$P = X^{-1}, \quad Q = P\tilde{Q}P.$$

Furthermore, the controller gain is given by

$$K = YX^{-1}.$$

Proof. According to Definition 1 and Theorem 1, system (1) is finite-time stabilizable with respect to (c_1, c_2, N) via state feedback controller $u(k) = Kx(k)$ if (35) and the following matrix inequality are admissible with respect to $\gamma \geq 1, P > 0, Q > 0$:

$$\begin{bmatrix} (d_M - d_m + 1)Q - \gamma P & 0 & (A_1 + BK)^T P & C_1^T P & 0 & \varepsilon_1 (N_1 + N_b K)^T & 0 & \varepsilon_2 N_3^T \\ * & -Q & A_d^T P & C_2^T P & 0 & \varepsilon_1 N_2^T & 0 & \varepsilon_2 N_4^T \\ * & * & -P & 0 & PM & 0 & 0 & 0 \\ * & * & * & -P & 0 & 0 & PM & 0 \\ * & * & * & * & -\varepsilon_1 I & 0 & 0 & 0 \\ * & * & * & * & * & -\varepsilon_1 I & 0 & 0 \\ * & * & * & * & * & * & -\varepsilon_2 I & 0 \\ * & * & * & * & * & * & * & -\varepsilon_2 I \end{bmatrix} < 0. \tag{36}$$

Setting $X = P^{-1}, \tilde{Q} = XQX^T, Y = KX$, pre and post multiplying (36) by $T = \text{diag}\{X, X, X, X, I, I, I, I\}$, the (34) can be obtained. The proof is completed.

4. Numerical example

In this section, two numerical examples are presented to show the application of the developed theory.

Example 1. Consider the uncertain stochastic system (1) with the following parameters:

$$A_1 = \begin{bmatrix} 0.01 & 0.02 \\ 0.02 & 0.04 \end{bmatrix}, \; A_d = \begin{bmatrix} 0.03 & 0.02 \\ 0.01 & 0.02 \end{bmatrix},$$

$$C_1 = \begin{bmatrix} 0.02 & 0.03 \\ 0.01 & 0.03 \end{bmatrix}, \; C_2 = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.04 \end{bmatrix},$$

$$M = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}, N_1 = N_3 = \begin{bmatrix} 0.02 & 0.02 \\ 0.02 & 0.02 \end{bmatrix},$$

$$N_2 = N_4 = \begin{bmatrix} 0.01 & 0.01 \\ 0.01 & 0.01 \end{bmatrix}, d(k) = 3 + \left|\sin(\frac{k\pi}{2})\right|,$$

Take $c_1 = 0.25, N = 10, \varepsilon_1 = 0.03, \varepsilon_2 = 0.04, \gamma = 1.5.$ Solving (8) and (9) leads to feasible solutions

$$P=\begin{bmatrix} 1.1699 & -0.0025 \\ -0.0025 & 1.1686 \end{bmatrix}, Q=\begin{bmatrix} 0.3898 & -0.0010 \\ -0.0010 & 0.3892 \end{bmatrix},$$

$c_2 = 48.27.$

According to Theorem 1, system (1) is finite-time stochastically stable with respect to $(0.25, 48.27, 10)$. Fig. 1 shows the simulation for state trajectories of the system (1) in Example 1.

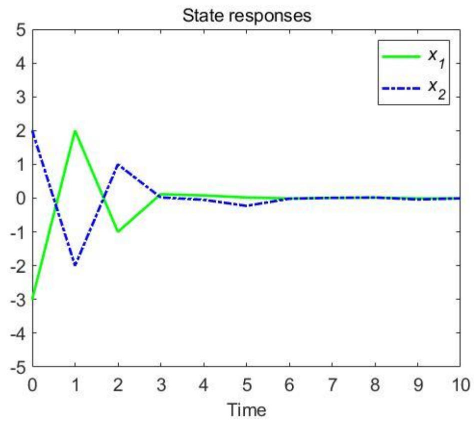

Fig.1. State response of the system in Example1

Example 2. Consider the uncertain stochastic system (33) with the following parameters:

$$A_1=\begin{bmatrix} 0.05 & 0.01 & 0.1 \\ 0.01 & 0.04 & 0.01 \\ 0.01 & 0.01 & 0.04 \end{bmatrix},\ A_d=\begin{bmatrix} 0.01 & 0.01 & 0.01 \\ 0.01 & 0.02 & 0.01 \\ 0.01 & 0.02 & 0.01 \end{bmatrix},$$

$$C_1=\begin{bmatrix} 0.01 & 0.01 & 0.02 \\ 0.02 & 0.02 & 0.01 \\ 0.01 & 0.02 & 0.01 \end{bmatrix},\ C_2=\begin{bmatrix} 0.01 & 0.01 & 0.01 \\ 0.01 & 0.02 & 0.01 \\ 0.01 & 0.01 & 0.01 \end{bmatrix},$$

$$B=\begin{bmatrix} 0.02 & 0.01 & 0.01 \\ 0.01 & 0.02 & 0.02 \\ 0.01 & 0.01 & 0.01 \end{bmatrix}, M=\begin{bmatrix} 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.01 \\ 0.01 & 0.01 & 0.01 \end{bmatrix},$$

$$N_1=N_2=N_3=N_4=\begin{bmatrix} 0.02 & 0.02 & 0.02 \\ 0.02 & 0.02 & 0.02 \\ 0.02 & 0.02 & 0.02 \end{bmatrix},$$

$$d(k)=3+\left|\sin(\frac{k\pi}{2})\right|, N_b=\begin{bmatrix} 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 \\ 0.03 & 0.03 & 0.03 \end{bmatrix}.$$

Take $c_1 = 0.5, N = 10, \varepsilon_1 = 0.02, \varepsilon_2 = 0.03, \gamma = 1.25.$ Solving (34) and (35) leads to feasible solutions

$$X=\begin{bmatrix} 1.0877 & -0.0032 & -0.0044 \\ -0.0032 & 1.0946 & -0.0021 \\ -0.0044 & -0.0021 & 1.0956 \end{bmatrix},$$

$$\tilde{Q}=\begin{bmatrix} 0.2869 & -0.0011 & -0.0015 \\ -0.0011 & 0.2893 & -0.0007 \\ -0.0015 & -0.0007 & 0.2897 \end{bmatrix},$$

$$Y=\begin{bmatrix} 0.9187 & -0.0025 & -0.0045 \\ -0.0025 & 0.9191 & -0.0039 \\ -0.0045 & -0.0039 & 0.9179 \end{bmatrix},$$

$c_2 = 17.23.$

The controller gain is

$$K=\begin{bmatrix} 0.8447 & 0.0002 & -0.0007 \\ 0.0002 & 0.8397 & -0.0020 \\ -0.0007 & -0.0020 & 0.8377 \end{bmatrix}.$$

According to Theorem 2, system (1) is finite-time stochastically stabilizable with respect to $(0.5, 17.23, 10)$. Fig. 2 shows the simulation results of the state trajectory of the closed-loop system (33).

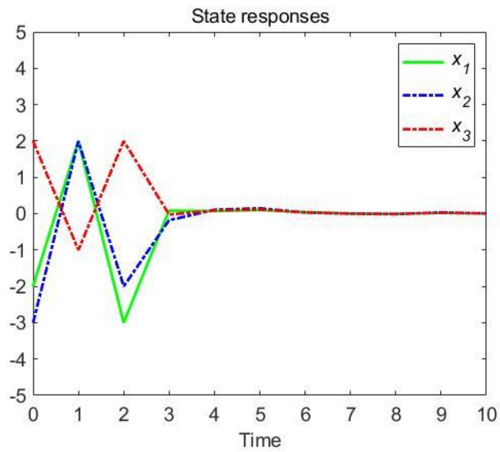

Fig.2. State response of the system (33)

5. Conclusions

In this paper, finite-time stochastically stability and stabilization problem has been investigated for a class of discrete time varying delay stochastic systems with uncertain. Sufficient conditions of finite-time stochastic stability have been given based on the Lyapunov-Krasovskii functional method. The criterion of finite-time stochastic stabilization for discrete-time stochastic system with time-varying delay is proposed. Finally, two numerical examples have been provided to show the applicability and less conservativeness of the presented results.

Acknowledgments

This work was supported by the Natural Science Foundation of Tianjin under Grant No. 18JCYBJC88000.

References:

[1] F. Amato, M. Ariola, P. Dorato, Finite time control of linear system subject to parametric uncertainties and disturbances, Automatica, Vol. 37, No. 9, 2001, pp. 1459-1463.

[2] F. Amato, M. Ariola, P. Dorate, Finite-time stabilization via dynamic output feedback, Automatica, Vol. 42 , No. 2, 2006, pp. 337-342.

[3] E. Moulay, W. Perruquetti, Finite time stability and stabilization of a class of continuous systems, Mathematical Analysis and Applications, Vol. 323, No. 2, 2006, pp. 1430-1443.

[4] S. B. Stojanovic, Finite-time stability of discrete-time systems with time-varying delay, Chemical Industry & Chemical Engineering Quarterly, Vol. 18, No. 4, 2012, pp. 525-533.

[5]T. Zhang, F. Deng, W. Zhang, Finite-time stability and stabilization of linear discrete time-varying stochastic systems, Journal of the Franklin Institute, Vol. 356, No. 3, 2019, pp. 1247-1267.

[6]M. J. Mirzaei, E. Aslmostafa, M. Asadollahi, M.A. Badamchizadeh, Robust adaptive finite-time stabilization control for a class of nonlinear switched systems based on finite-time disturbance observer, Journal of the Franklin Institute, Vol. 358, No. 7, 2021, pp. 3332-3352.

[7] S. Khoo, J. Yin, Z. Man, X. Yu, Finite-time stabilization of stochastic nonlinear systems in strict-feedback form, Automatica, Vol. 49, No. 5 , 2013, pp. 1403-1410.

[8] X. Li, X. Lin, S. Li, Y. Zou, Finite-time stability of switched nonlinear systems with fifinite time unstable subsystems, Journal of the Franklin Institute, Vol. 352, No. 3, 2015, pp. 1192-1214.

[9] X. Lin, H. Du, S. Li, Y. Zou, Finite-time stability and finite-time weighted L2-gain analysis for switched systems with time-varying delay, IET Control Theory & Applications, Vol. 7, No. 7, 2013, pp. 1058-1069.

[10]H. Wang, Q. Zhu, Finite-time stabilization of high-order stochastic nonlinear systems in strict-feedback form, Automatica, Vol. 54, pp. 2015, 284-291.

[11] G. Chen, Y. Yang, New necessary and sufficient conditions for finite-time stability of impulsive switched linear time-varying systems, IET Control Theory Appl, Vol. 12, 2018, pp. 140-148.

[12] S.B. Stojanovic, Robust finite-time stability of discrete time systems with interval time-varying delay and nonlinear perturbations, Journal of the Franklin Institute, Vol. 354 , No. 11, 2017, pp. 4549-4572.

[13] Z. Zuo, H. Li, Y. Wang, New criterion for finite-time stability of linear discrete-time systems with time-varying delay, Journal of the Franklin Institute, Vol. 350, No. 9, 2013, pp. 2745-2756.

[14] M. Luo, S. Zhong, R. Wang, W. Kang, Robust stability analysis for discrete-time stochastic neural networks systems with time-varying delays, Applied Mathematics and Computation , Vol. 209, No. 2, 2009, pp. 305-313.

[15] A. Arunkumar, R. Sakthivel, K. Mathiyalagan, S. Marshal Anthoni, Robust stability criteria for discrete-time switched neural networks with various activation functions, Applied Mathematic and Computation, Vol. 218, No. 22, 2012, pp. 10803-10816.

[16] Z. Zuo, Y. Liu, Y. Wang, H. Li, Finite-time stochastic stability and stabilisation of linear discrete-time Markovian jump systems with partly unknown transition probabilities, IET Control Theory & Application, Vol. 6, No. 10, 2012, pp. 1522-1526.

[17] V. Kumara, M. Djemai, M. Defoort, M. Malik, Finite-time stability and stabilization results for switched impulsive dynamical systems on time scales, Journal of the Franklin Institute, Vol. 358, No. 1, 2021, pp. 674-698.

[18] X. Li, X. Yang, S. Song, Lyapunov conditions for finite-time stability of time-varying time-delay systems, Automatica, Vol. 103, 2019, pp. 135-140.

[19] E. Moradi, Finite time stabilization of time-delay nonlinear systems with uncertainty and time-varying delay, Journal of Control, Vol. 14, No. 2, 2020, pp. 79-87.

[20] Y. Liu, Z. Wang, X. Liu, Robust H_∞ filtering for discrete nonlinear stochastic systems with time-varying delay, J. Math. Anal. Appl, Vol. 341, No.1, 2008, pp. 318-336.

Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

The authors equally contributed in the present research, at all stages from the formulation of the problem to the final findings and solution.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

This work was supported by the Natural Science Foundation of Tianjin under Grant No. 18JCYBJC88000.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.