Some Topological Properties on The Cartesian Product of Fuzzy b-Metric Space

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Abstract: Abstract: The aim of this paper is to introduce the concept of the Cartesian product of two fuzzy b-metric spaces with related properties. We have introduced the property that the Cartesian product of two complete fuzzy b-metric spaces is a complete fuzzy b-metric space in terms of the product t-norm $(.)$*, as well as the minimum t-norm* $(\wedge)$*. The properties of convergent and Cauchy sequence are introduced by using this concept.*

Key-Words:Fuzzy b-metric space, Convergent sequence, Cauchy sequence, Cartesian product,Complete

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1 Introduction

There are numerous ways to generalize the idea of metric spaces, one of them is a b-metric space. Now a days The theory of metric space has become an emerging field of research and one of those methods is the fuzzy idea. With the study in new horizon of research, Zadeh [5] introduced the concept of fuzzy sets. Many eminent scholars have introduced the applications of fuzzy set theory to the terminology used in topology and analysis. It is well recognized that scientists and mathematicians can benefit by using the concept of fuzzy metric space as a generalization. Several researchers have introduced fuzzy metric spaces using various methods. In general it is not possible to measure the exact distance between any two places precisely. Thus we conclude that while measuring the same distance between two places in different times, we will get the different results. This situation can be handled by two ways probabilistic and statistical approach. But by using the probabilistic approach it uses the idea of distribution function instead non-negative real numbers. As the uncertainty in the distance between two points is due to fuzziness instead of randomness. Consequently, by using the continuous t-norm, fuzzy metric space was defined by many researchers. Latter on it was Updated by George and Veeramani [1].

In 2012, Sedghi and Shobe [11] found the new idea with common fixed point theorem in fuzzy b-metric space. Hussain et al.[4] established the concept of a fuzzy b-metric space and obtained various fixed point theorems for this type of space through their publication of an article in 2015. Although Hussain's is formally correct, it is mathematically unjustified. Nadaban [10] introduced some of the topological aspects of a fuzzy b-metric space and presented the concept of a fuzzy b-metric space and also explored some of its topological properties. The contraction mapping in partial b-metric space was introduced by Amar et al. [3] and has certain applications. On the other hand, the study on product spaces in the probabilistic framework was initiated by Istratescu, Vaduva, Mohd. Rafi and M.S.M Noorani subsequently by Egbert, Alsina [1] and Schweizer [2]. Recently, Lafuerza-Guillen [13] has studied finite products of probabilistic normed spaces and proved some interesting results.The concepts of product of probabilistic metric (normed) spaces studied by Egbert (Lafuerza -Guillen). Similarly, The Cartesian product of two fuzzy metric spaces was introduced by Jehad R. Kider [8] by publishing his paper in 2011. Later on in 2022, Mayada et al.[6] developed the new properties in fuzzy b-metric space by introducing two continuous t-norms on the completeness of the characterization.

The objective of our research is to discuss the different topological properties in fuzzy b-metric by using convergent sequence, Cauchy sequence and completeness to introduce the concept of Cartesian product in fuzzy b-metric space. Also other inherited significant properties related to fuzzy b-metric with Cartesian product space are examined.

2 Preliminaries

In 1942, the operation of t-norm was introduced by K. Menger and by using the concept of continuity in these operations and in 1960, B. Schweizer and Sklar introduced the operation of t-norm and using the concept of continuity.

Definition 2.1 [2] A binary operation $* : [0,1] \times [0,1] \rightarrow [0,1]$ is continuous triangular norm (t-norm) if for all $p, q, r, s \in [0,1]$, the following conditions are satisfied
(T1) $p * 1 = p$;
(T2) $p * q = q * p$;
(T3) If $p \leq r$ and $q \leq s$ then $p * q \leq r * s$;
(T4) $p * (q * r) = (p * q) * r$
The basic examples of continuous t-norm are: $T_p(p,q) = p.q$ $(usual\ multiplication\ in[0,1])$, $T_m(p,q) = min\{p,q\}$ and $T_L(p,q) = max(p+q-1, 0)$ (the Lukasiewicz t-norm).
Definition 2.2 [1] A triplet $(U, F, *)$ is known as a fuzzy metric space if U is any set, $*$ is a continuous t-norm and F is a fuzzy set defined on $U \times U \times (0,\infty) \rightarrow [0,1]$ if it satisfies the properties given as below, for all $u, v, w \in U$ and $t, s > 0$
(FM-1) $F(u,v,t) > 0$,
(FM-2)$F(u,v,t) = 1 \iff u = v$
(FM-3)$F(u,v,t) = F(v,u,t)$,
(FM-4)$F(u,v,t) * F(v,w,s) \leq F(u,v,t+s)$ for all $t, s > 0$,
(FM-5)$F(u,v,.) : (0,\infty) \rightarrow [0,1]$ is continuous
The degree of nearness between u and v with respect to $t > 0$ is denoted by $F(u,v,t)$.
Example 2.3 [1] Let U be a non empty set and d be a metric defined on U and $*$ is continuous t-norm defined as $p * q = p.q$ for all $p, q \in [0,1]$ then the relation defined by

$$F(u,v,t) = \frac{t}{t + d(u,v)}$$

is fuzzy metric in $[0,1]$ and $(U, F, *)$ is a fuzzy metric space.
In 2016, Nadaban [10] explored the idea of fuzzy b-metric space to generalize the notion of the fuzzy metric spaces introduced by Kramosil and Michalek.
Definition 2.4 [10] Let U be a nonempty set, $*$ is a continuous t-norm and F be a fuzzy set. Then a mapping F: $U \times U \times \mathbb{R}^+ \rightarrow [0,1]$ is called a fuzzy b-metric space. If there exists $k \geq 1$ such that the following properties are fulfilled
for all $u, v, w \in U$ and $t, s > 0$
(Fb-i) $F(u,v,t) > 0$;
(Fb-ii) $F(u,v,t) = 1$ for all $t > 0$ if and only if $u = v$;
(Fb-iii) $F(u,v,t) = F(v,u,t)$;
(Fb-iv) $F(u,v,t) * F(v,w,s) \leq F(u,w,k(t+s))$ for all $t, s > 0$;
(Fb-v) $F(u,v,.) : (0,\infty) \rightarrow [0,1]$ is left continuous ;
(Fb-vi) $\lim_{t\rightarrow\infty} F(u,v,t) = 1$
The class of fuzzy b-metric space is larger than the class of fuzzy metric space. In fuzzy b-metric space if we put $k = 1$ then it becomes a fuzzy metric space.

Definition 2.5 [1] Let $(U, F, *)$ be a fuzzy b-metric space. A sequence $\{u_n\}$ in U is said to be convergent in U if
$\lim_{n\rightarrow\infty} F(u_n, u, t) = 1$ for each $t > 0$.
where $\lim_{n\rightarrow\infty} u_n = u$
Definition 2.6 [1] A sequence $\{u_n\}$ in U is said to be Cauchy sequence in U if
$\lim_{n\rightarrow\infty} F(u_n, u_{m+n}, t) = 1$ where $t > 0$ and $m, n > 0$.
If every Cauchy sequence is convergent in a fuzzy b-metric space the it is called the complete fuzzy b-metric space.
An implementation of our findings in fuzzy b-metric space is obtained from the constructive discussion as below.
Example 2.7 [1] Let us define a mapping F from $U \times U \times \mathbb{R}^+$ to the unit interval $[0,1]$ as

$$F(u,v,t) = \begin{cases} \frac{t-d(u,v)}{t+d(u,v)} & \text{for } t > d(u,v) \\ 0 & \text{for } t \leq d(u,v) \end{cases}$$

Then $(U, F, *)$ is an fuzzy b-metric space, where $p * q = \min\{p,q\}$ d be a metric on U.
By the above definition of fuzzy b-metric space, the properties **(i)**, **(ii)**, as well **(iii)** are obviously fulfilled. So, we will fulfill the property **(iv)** where $u, v, w \in U$ and $t, s \in (0,\infty)$, the following cases follow:

a) If $t \leq d(u,v)$ or $s \leq d(v,w)$ or both, then

$$F(u,w,k(t+s)) \geq F(u,v,t) * F(v,w,s)$$

is satisfied obviously.
(b) If $t > d(u,v)$ and $s > d(v,w)$, then

$$\begin{aligned} & F(u,w,k(t+s)) \\ & \geq \frac{k(t+s) - d(u,w)}{k(t+s) + d(u,w)} \\ & \geq \frac{k(t+s) - k[d(u,v) + d(v,w)]}{k(t+s) + k[d(u,v) + d(v,w)]} \\ & \geq \frac{(t+s) - d(u,v) - d(v,w)}{(t+s) + d(u,v) + d(v,w)} \\ & \geq \min\left\{\frac{t - d(u,v)}{t + d(u,v)}, \frac{s - d(v,w)}{s + d(v,w)}\right\} \\ & = F(u,v,t) * F(v,w,s). \end{aligned}$$

It means that $F(u,w,k(t+s)) \geq F(u,v,t) * F(v,w,s)$.
Now to fulfill the property (v), assume that $\{t_n\}$ is a sequence in $[0,\infty)$ such that $\{t_n\}$ converges

to t. Then, for all $u, v \in U$,

$$\begin{aligned}\lim_{n\to\infty} F(u,v,t_n) &= \lim_{n\to\infty} \frac{t_n - d(u,v)}{t_n + d(u,v)} \\ &= \frac{t - d(u,v)}{t + d(u,v)} \\ &= F(u,v,t).\end{aligned}$$

So $F(u,v,t_n)$ converges to $F(u,v,t)$.
Hence $F(u,v,\cdot) : [0,\infty) \to [0,1]$ is left-continuous, and

$$\lim_{t\to\infty} F(u,v,t) = \lim_{t\to\infty} \frac{t - d(u,v)}{t + d(u,v)} = 1.$$

Hence the properties of fuzzy b-metric space are satisfied. So $(U, F, *)$ is a fuzzy b-metric space.
Now we will give the example of convergent sequence and Cauchy sequence in fuzzy b-metric space defined as:
Example 2.8 Let us define a mapping F from $U \times U \times \mathbb{R}^+$ to the unit interval $[0,1]$ as

$$F(u,v,t) = \begin{cases} e^{-\dfrac{(u-v)^2}{t}}, & \text{if } t > 0, \\ 0, & \text{if } t = 0 \end{cases}$$

If $U = \mathbb{R}^+$ and there exists $k \geq 1$. Then $(U, F, *)$ is a fuzzy b-metric space.
Example 2.9 Assume that $\{u_n\}$ in U where $\{u_n\} = \dfrac{1}{n}$ for all $n \in N$, and the fuzzy b-metric space is defined as in the above example (2.8), then clearly $\{u_n\}$ is converges to 0, which is shown as follow:

$$\begin{aligned}\lim_{n\to\infty} F(u_n, 0, t) &= \lim_{n\to\infty} e^{\dfrac{-(u_n - 0)^2}{t}} \\ &= \lim_{n\to\infty} e^{\dfrac{-(u_n)^2}{t}} \\ &= \lim_{n\to\infty} e^{\dfrac{\frac{-1}{n^2}}{t}} \\ \lim_{n\to\infty} e^{\dfrac{-1}{n^2 t}} &= 1.\end{aligned}$$

Hence the sequence $\{u_n\}$ is convergent in fuzzy b-metric space.
Example 2.10 Consider a fuzzy b-metric space $(U, F, *)$ defined as above example (2.8) and a sequence $\{u_n\}$ with
$\{u_n\} = \dfrac{1}{n}$ for all $n \in \mathbb{N}$.

Now for all $m \in \mathbb{N}$, we have

$$\begin{aligned}\lim_{n\to\infty} F(u_n, u_{m+n}, t) &= \lim_{n\to\infty} e^{\dfrac{-(u_n - u_{m+n})^2}{t}} \\ &= \lim_{n\to\infty} e^{\dfrac{-(\frac{1}{n} - \frac{1}{m+n})^2}{t}} \\ &= 1\end{aligned}$$

Hence $\{u_n\}$ is a Cauchy sequence in fuzzy b-metric space.

3 Cartesian Product of Two Fuzzy b-Metric Spaces

Now, we use the concept of the Cartesian product of two fuzzy b-metric spaces, then we prove that the Cartesian product of two fuzzy b-metric spaces is also fuzzy b-metric space.
Finally, we prove the completeness of the Cartesian product of two complete fuzzy b-metric spaces together with the properties of limit point and Cauchy sequence.
Definition 3.1 [9] Let $(U_1, F_1, *)$ and $(U_2, F_2, *)$ be two fuzzy b-metric spaces. The Cartesian product of $(U_1, F_1, *)$ and $(U_2, F_2, *)$ is the product space $(U_1 \times U_2, F, *)$ where $(U_1 \times U_2)$ is the Cartesian product of U_1 and U_2 and F is a mapping from $((U_1 \times U_2) \times (0,\infty)) \times ((U_1 \times U_2) \times (0,\infty)) \to [0,1]$ and

$$F((u_1,u_2),(v_1,v_2),t) = F_1(u_1,v_1,t) \cdot F_2(u_2,v_2,t), \quad \forall t > 0$$

where $(u_1,u_2),(v_1,v_2) \in U_1 \times U_2$ and $*$ is a continuous t-norm and $F = F_1 \cdot F_2$. Then the product space $(U_1 \times U_2, F, *)$ is known as the Cartesian product of two fuzzy b-metric spaces.
Theorem 3.2 Let $(U_1, F_1, *)$ and $(U_2, F_2, *)$ be any two fuzzy b-metric spaces, if there exists areal number $k \geq 1$, where $(u_1,u_2),(v_1,v_2) \in U_1 \times U_2$, such that

$$F((u_1,u_2),(v_1,v_2),t) = F_1(u_1,v_1,t) \cdot F_2(u_2,v_2,t).$$

Then $(U_1 \times U_2, F, *)$ is a fuzzy b-metric space.
Proof: In order to complete the proof of the theorem, the following properties should be satisfied.

(i) Since $F_1(u_1,v_1,t) > 0$ and $F_2(u_2,v_2,t) > 0$,
Which gives $F_1(u_1,v_1,t) * F_2(u_2,v_2,t) > 0$.
So, $F((u_1,u_2),(v_1,v_2),t) > 0$.

(ii) Assume that $(u_1,u_2) = (v_1,v_2)$.
Which gives $u_1 = v_1$ and $u_2 = v_2$. Hence, for all $t > 0$, we have $F_1(u_1,v_1,t) = 1$ and

$F_2(u_2, v_2, t) = 1$.
It follows that $F((u_1, u_2), (v_1, v_2), t) = 1$.
Conversely, let us assume that $F((u_1, u_2), (v_1, v_2), t) = 1$.
Which gives $F_1(u_1, v_1, t) * F_2(u_2, v_2, t) = 1$.
Since $0 < F_1(u_1, v_1, t) \leq 1$ and $0 < F_2(u_2, v_2, t) \leq 1$,
it gives $F_1(u_1, v_1, t) = 1$ and $F_2(u_2, v_2, t) = 1$.
Hence, $u_1 = v_1$ and $u_2 = v_2$.
So $(u_1, u_2) = (v_1, v_2)$.

(iii) To show $F((u_1, u_2), (v_1, v_2), t) = F((v_1, v_2), (u_1, u_2), t)$, we note that

$$F_1(u_1, v_1, t) = F_1(v_1, u_1, t), \quad F_2(u_2, v_2, t) = F_2(v_2, u_2, t).$$

It follows that for all $(u_1, u_2), (v_1, v_2) \in U_1 \times U_2$ and $t > 0$,

$$F((u_1, u_2), (v_1, v_2), t) = F((v_1, v_2), (u_1, u_2), t).$$

(iv) Since $(U_1, F_1, *)$ and $(U_2, F_2, *)$ are two fuzzy b-metric spaces, we have

$$\begin{aligned} F_1(u_1, w_1, k(t+s) &\geq F_1(u_1, v_1, t) \\ &\quad * F_1(v_1, w_1, s), \\ &F_2(u_2, w_2, k(t+s)) \\ &\geq F_2(u_2, v_2, t) \\ &\quad * F_2(v_2, w_2, s) \end{aligned}$$

for all $(u_1, u_2), (v_1, v_2), (w_1, w_2) \in U_1 \times U_2$ and $t, s > 0$.

$$\begin{aligned} &F((u_1, u_2), (w_1, w_2), k(t+s)) \\ &= F_1(u_1, w_1, k(t+s)) \\ &\cdot F_2(u_2, w_2, k(t+s)) \\ &\geq [F_1(u_1, v_1, t) \\ &* F_1(v_1, w_1, s)].[F_2(u_2, v_2, t) \\ &* F_2(v_2, w_2, s)] \\ &\geq [F_1(u_1, v_1, t) \cdot F_2(u_2, v_2, t)] \\ &* [F_1(v_1, w_1, s) \cdot F_2(v_2, w_2, s)] \\ &\geq F((u_1, u_2), (v_1, v_2), t) \\ &* F((v_1, v_2), (w_1, w_2), s). \end{aligned}$$

(v) Since $F_1(u_1, v_1, t)$ and $F_2(u_2, v_2, t)$ are non-decreasing functions on $\mathbb{R}^+$.
So $\lim_{t\to\infty} F_1(u_1, v_1, t) = 1$ and $\lim_{t\to\infty} F_2(u_2, v_2, t) = 1$.
Then $\lim_{t\to\infty} F((u_1, v_1)(u_2, v_2), t)$
$= \lim_{t\to\infty} F_1(u_1, v_1, t). \lim_{t\to\infty} F_2(u_2, v_2, t)$
$= 1.1 = 1$.

$$F((u_1, u_2), (v_1, v_2), t) = F_1(u_1, v_1, t) \cdot F_2(u_2, v_2, t)$$

is also continuous.
Hence $(U_1 \times U_2, F, *)$ is a fuzzy b-metric space.

Theorem 3.3 If $\{u_n\}$ is a sequence in fuzzy b-metric space $(U_1, F_1, *)$ converging to u in U_1, and $\{v_n\}$ is a sequence in the fuzzy b-metric space $(U_2, F_2, *)$ converging to v in U_2, then $\{(u_n, v_n)\}$ is a sequence in the fuzzy b-metric space $(U_1 \times U_2, F)$ converging to (u, v) in $U_1 \times U_2$, where $F = F_1 \cdot F_2$.
Proof: By since, $(U_1 \times U_2, F, *)$ is a fuzzy b-metric space. Now for each $t > 0$,

$$\begin{aligned} &\lim_{n\to\infty} F((u_n, v_n), (u, v), t) \\ &= \left[\lim_{n\to\infty} F_1(u_n, u, t)\right] \\ &\cdot \left[\lim_{n\to\infty} F_2(v_n, v, t)\right] \\ &= 1.1 = 1. \end{aligned}$$

Using the above theorem 3.2. Hence the pair of sequence $\{(u_n, v_n)\}$ converges to (u, v).
Theorem 3.4 Let us assume that $\{u_n\}$ be a Cauchy sequence in a fuzzy b-metric space $(U_1, F_1, *)$ and the sequence $\{v_n\}$ is Cauchy in a fuzzy b-metric space $(U_2, F_2, *)$, where $*$ is a continuous t-norm. Then $\{(u_n, v_n)\}$ be the Cauchy sequence in the product space $(U_1 \times U_2, F, *)$.
Proof: Since $(U_1 \times U_2, F, *)$ is a fuzzy b-metric space. As our supposition $\{u_n\}$ and $\{v_n\}$ are two Cauchy sequences. So for each $t > 0$ and $m > 0$,
$\lim_{n\to\infty} F((u_{n+m}, v_{n+m}), (u_n, v_n), t)$
$= [\lim_{n\to\infty} F_1((u_{n+m} u_n, t) \cdot \lim_{n\to\infty} F_2((v_{n+m} v_n, t)]$
Hence $\lim_{n\to\infty} F_1((u_{n+m} u_n, t) = 1$ and $\lim_{n\to\infty} F_2((v_{n+m} v_n, t) = 1$
Thus $\{(u_n, v_n)\}$ is a Cauchy sequence in $(U_1 \times U_2, F, *)$.
Theorem 3.5 Let $(U_1, F_1, *)$ and $(U_2, F_2, *)$ are any two fuzzy b-metric spaces. Then $(U_1 \times U_2, F, *)$ is complete if and only if $(U_1, F_1, *)$ and $(U_2, F_2, *)$ are complete.
proof: Assume that $(U_1, F_1, *)$ and $(U_2, F_2, *)$ are complete fuzzy b-metric spaces.
Suppose $\{(u_n, v_n)\}$ be a Cauchy sequence in $U_1 \times U_2$.
that is for each $t > 0$ and $m > 0$,
$\lim_{n\to\infty} F((u_{n+m}, v_{n+m}), (u_n, v_n), t)$
$= [\lim_{n\to\infty} F_1((u_{n+m}, u_n, t) \cdot \lim_{n\to\infty} F_2((v_{n+m}, v_n, t)]$
Hence
$\lim_{n\to\infty} F_1((u_{n+m}, u_n, t) = 1$ and

$\lim_{n\to\infty} F_2((v_{n+m}, v_n, t) = 1$
Therefore $\{u_n\}$ is a Cauchy sequence in (U_1, F_1*) and $\{v_n\}$ is a Cauchy sequence in $(U_2, F_2, *)$.
But $(U_1, F_1, *)$ and $(U_2, F_2, *)$ are complete fuzzy b-metric spaces, so there exists $u \in U_1$ and $v \in U_2$ such that for each $t > 0$
$\lim_{n\to\infty} F_1(u_n, u, t) = 1$ and $\lim_{n\to\infty} F_2(v_n, v, t) = 1$.
Hence $\{(u_n, v_n)\}$ converges to (u, v) in $U_1 \times U_2$. Therefore $(U_1 \times U_2, F, *)$ is a complete fuzzy b-metric space.
Conversely, suppose that $(U_1 \times U_2, M, *)$ is complete.
We will show that $(U_1, F_1, *)$ and $(U_2, F_2, *)$ are complete.
Let $\{u_n\}$ and $\{v_n\}$ be Cauchy sequences in $(U_1, F_1, *)$ and $(U_2, F_2, *)$, respectively.
Then $F_1(u_{m+n}, u_n, t)$ converges to 1 and $F_2(v_{m+n}, v_n, t)$ converges to 1 for each $t > 0$ and $m > 0$. It follows that
$\lim_{n\to\infty} F((u_{n+m}, v_{n+m}), (u_n, v_n), t)$
$= [\lim_{n\to\infty} F_1((u_{n+m}, u_n, t) \cdot \lim_{n\to\infty} F_2((v_{n+m}, v_n, t)]$
converges to 1.
Thus $\{(u_n, v_n)\}$ is a Cauchy sequence in $U_1 \times U_2$. Since $(U_1 \times U_2, F, *)$ is complete, there exists $(u, v) \in U_1 \times U_2$ such that $F((u_n, v_n), (u, v), t)$ converges to 1, as lim $n \to \infty$.
Clearly, $F_1(u_n, u, t)$ converges to 1 and $F_2(v_n, v, t)$ converges to 1.
Hence, $(U_1, F_1, *)$ and $(U_2, F_2, *)$ are complete.
Theorem 3.6 Let u be a limit point of $\{u_n\}$ in a fuzzy b-metric space $(U_1, F_1, *)$ and u' be a limit point of $\{u'_n\}$ in a fuzzy b-metric space $(U_2, F_2, *)$. Where $*$ is a continuous t-norm, and (u, u') is the point of limit (u_n, u'_n) in the fuzzy b-metric space $(U_1 \times U_2, F, *)$.
Proof: The space $(U_1 \times U_2, F, *)$ is fuzzy b-metric space then by above Theorem (3.5). Since u be a limit of (u_n), thus for all $t > 0$, $\lim_{n\to\infty} F_1(u_n, u, t) = 1$ and since u' be a limit of (u'_n), thus $\lim_{n\to\infty} F_2(u'_n, u', t) = 1$. Now for all $t > 0$,

$$\lim_{n\to\infty} F((u_n, u'_n), (u, u'), t)$$
$$= \lim_{n\to\infty} F_1(u_n, u, t).$$
$$\lim_{n\to\infty} F_2(u'_n, u', t)$$
$$= 1.1 = 1.$$

Hence $F((u_n, u'_n), (u, u'), t) = 1$, which gives (u, u') is the limit point of (u_n, u'_n).

4 Completeness of $U_1 \times U_2$ with $F = F_1 \wedge F_2$

Here we introduce the fuzzy b- metric space in terms of the minimum t-norm $\wedge$. All the properties of fuzzy b-metric space are remained same, only the triangle inequality is redefined as:
$F(u, v, k(s + t)) \geq F(u, w, s) \wedge F(w, v, t)$ for each $t, s > 0$, where $\wedge$ is the minimum of $F(u, v, s)$ and $F(w, v, t)$.
Then we will show that the Cartesian product of two fuzzy b-metric spaces is a fuzzy b- metric space and the properties of convergent sequence and Cauchy sequence in terms of the ordered pair in fuzzy b-metric space are established by using the the minimum triangular norm $\wedge$.
Theorem 4.1 If $(U_1, F_1, *)$ and $(U_2, F_2, *)$ are fuzzy b-metric spaces. If there exists $k \geq 1$, then $(U_1 \times U_2, F, *)$ is a fuzzy metric space by defining

$$F((u_1, v_1), (u_2, v_2), t) = F_1(u_1, u_2, t) \wedge F_2(v_1, v_2, t).$$

Proof: Let $(u_1, v_1), (v_2, y_2), (u_3, v_3) \in U_1 \times U_2$:
(i) Let $t \geq 0$, we have $F_1(u_1, u_2, t) = 0$ and $F_2(v_1, v_2, t) = 0$. Hence,

$$F((u_1, v_1), (u_2, v_2), t) = 0.$$

$F_1(u_1, u_2, t) = 1$ for each $t > 0 \iff u_1 = u_2$, and $F_2(v_1, v_2, t) = 1$ for each $t > 0 \iff v_1 = v_2$.
Together with,

$$[F_1(u_1, u_2, t) \wedge F_2(v_1, v_2, t)] = 1$$

for $t > 0 \iff u_1 = u_2$ and $v_1 = v_2$. That is,

$$F((u_1, v_1), (u_2, v_2), t) = 1$$

$\iff (u_1, v_1) = (u_2, v_2)$.
(iii) $F_1(u_1, u_2, t) = F_1(u_2, u_1, t)$ and
$F_2(v_1, v_2, t) = F_2(v_2, v_1, t)$ Now for each $t > 0$,

$$\begin{aligned} F((u_1, v_1), (u_2, v_2), t) &= [F_1(u_1, u_2, t) \wedge F_2(v_1, v_2, t)] \\ &= [F_1(u_2, u_1, t) \wedge F_2(v_2, v_1, t)] \\ &= F((u_2, v_2), (u_1, v_1), t). \end{aligned}$$

(iv) $F_1(u_1, u_2, k(s + t)) \geq [F_1(u_1, u_3, s) * F_1(u_3, u_2, t)]$ for each $s, t > 0$.
Also, $F_2(v_1, v_2, k(s + t)) \geq [F_2(v_1, v_3, s) * F_2(v_3, v_2, t)]$ for each $s, t > 0$. Now for each $t > 0$,
$F((u_1, v_1), (u_2, v_2), k(s + t))$
$= [F_1(u_1, u_2, k(s + t)]$
$\wedge F_2(v_1, v_2, k(s + t))$
$\geq [F_1(u_1, u_3, s) * F_1(u_3, u_2, t)$
$\wedge F_2(v_1, v_3, s) * F_2(v_3, v_2, t)]$
$\geq [F_1(u_1, u_3, s) \wedge F_2(v_1, v_3, s)] * [F_1(u_3, u_2, t) \wedge F_2(v_3, v_2, t)]$
$\geq [F((u_1, v_1), (u_3, v_3), s) * F((u_3, v_3), (u_2, v_2), t)]$.

(v) Since $F_1(u_1, v_1, t)$ and $F_2(u_2, v_2, t)$ are non-decreasing functions on $\mathbb{R}^+$.
So $\lim_{t\to\infty} F_1(u_1, v_1, t) = 1$ and $\lim_{t\to\infty} F_2(u_2, v_2, t) = 1$.
Now,

$$\begin{aligned} F((u_1, v_1), (u_2, v_2), t) &= [F_1(u_1, u_2, t) \wedge F_2(v_1, v_2, t)] \\ &\leq [F_1(u_1, u_2, t) \wedge F_2(v_1, v_2, t)] \\ &\leq F((u_1, v_1), (u_2, v_2), t). \end{aligned}$$

and

$$\begin{aligned} &\lim_{t\to\infty} F((u_1, v_1), (u_2, v_2), t) \\ &\qquad = [\lim_{t\to\infty} F_1(u_1, u_2, t)] \wedge \\ &\qquad [\lim_{t\to\infty} F_2(v_1, v_2, t)] = 1. \end{aligned}$$

Thus $(U_1 \times U_2, F, *)$ is a fuzzy b-metric space.
Corollary 4.2: If $\{u_n\}$ is a sequence in the fuzzy b-metric space $(U_1, F_1, *)$ converging to $u \in U_1$, and $\{v_n\}$ is a sequence in the fuzzy b-metric space $(U_2, F_2, *)$ converging to $v \in U_2$, then $\{(u_n, v_n)\}$ is a sequence in the fuzzy metric space $(U_1 \times U_2, F, *)$ converging to (u, v), where $F = F_1 \wedge F_2$.
Proof: By above Theorem 4.1, $(U_1 \times U_2, F, *)$ is a fuzzy b-metric space. Now for each $t > 0$,

$$\lim_{n\to\infty} F((u_n, v_n), (u, v), t)$$

$$= [\lim_{n\to\infty} F_1(u_n, u, t)]$$

$$\wedge[\lim_{n\to\infty} F_2(v_n, v, t)] = 1 \wedge 1 = 1.$$

Hence $\{(u_n, v_n)\}$ converges to (u, v).
Corollary 4.3: If $\{u_n\}$ is a Cauchy sequence in the fuzzy b-metric space $(U_1, F_1, *)$ and $\{v_n\}$ is a Cauchy sequence in the fuzzy b-metric space $(U_2, F_2, *)$, then $\{(u_n, v_n)\}$ is a Cauchy sequence in the fuzzy b-metric space $(U_1 \times U_2, F, *)$, where $F = F_1 \wedge F_2$.
Example 4.5 Let $(U_1, F, *)$ and $(U_2, F, *)$ be two fuzzy b-metric spaces and assume $(U_1 \times U_2, d)$ be their product space, where

$$d(a, b) = \max\{U(u_1, u_2), U_2(v_1, v_2)\}$$

for each $a = (u_1, v_1)$ and $b = (u_2, v_2)$ in $U_1 \times U_2$. Define $p \wedge q = \min(p, q)$ for all $p, q \in [0, 1]$ and assume

$$U_1(a, b, t) = \frac{t}{t + d(a, b)}.$$

Then $(U_1 \times U_2, F, *)$ is a $\wedge$-product of (U, dU_1) and (U_2, dU_2).

Proof: Since we have,

$$\begin{aligned} F(a, bt) =& \frac{t}{t + d(a, b)} \\ =& \frac{t}{t + \max\{dU_1(u_1, u_2), dU_2(v_1, v_2)\}} \\ =& \min\left(\frac{t}{t + dU_1(u_1, u_2)}, \frac{t}{t + dU_1(v_1, v_2)}\right) \end{aligned}$$

$$= \left(\frac{t}{t + dU_1(u_1, u_2)}\right) \wedge \left(\frac{t}{t + du_2(v_1, v_2)}\right).$$

Thus, $M_d(a, b, t) = FdU_1 \wedge FdU_2$.
Hence, $(U_1 \times U_2, F, *)$ is a $\wedge$-product of (U, dU_1) and (U_2, dU_2).

5 Conclusion

By introducing the idea of Cartesian product of two fuzzy b-metric spaces with suitable properties, we have proved that the Cartesian product of two complete fuzzy b-metric spaces is again a fuzzy b-metric space under the product t-norm as well as the minimum t-norm. Also we have proved the properties of convergence sequence and Cauchy sequence in terms of the Cartesian product in fuzzy b-metric space and we have presented some examples to verify the definition of fuzzy b-metric space, convergent sequence and Cauchy sequence as well. Further research can be done by connecting the fixed point theory in fuzzy b-metric space in terms of the Cartesian product. Further more, many topological properties can be connected with this topic.

Conflict of Interest

The authors have no conflicts of interest to declare that are relevant to the content of this article.

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