# The generalization of Fourier-transform and the Peter-Weyl theorem

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*Abstract:* - This article is devoted to the generalization of the Fourier transform and harmonic analysis on compact Hausdorff groups, we construct the Fourier-Stieltjes calculus, which is associated with the semigroups on the Hilbert space. We obtain that let $U_1 : L^1(G) \to L(H,H)$ be a nondegenerate unitary representation then there exists a unique representation $U : G \to U(H)$ such that $U_1 = U_{st}$ . Also, we establish that assume $U : G \to U(H)$ is a unitary representation of the group $G$ and assume $U_{st} : L^1(G) \to L(H,H)$ is a unitary representation of functional space $L^1(G)$ then there is a mapping $\Upsilon : U \to U_{st}$, which is a bijection.

*Key-Words:* - Fourier transform, Peter-Weyl theorem, Banach algebra, compact group.

Received: September 7, 2021. Revised: May 11, 2022. Accepted: June 9, 2022. Published: July 5, 2022.

## 1 Introduction

The article is dedicated to the Fourier theory on the compact Hausdorff topological groups, the example of such a group $G$ is the circle group $S^1 = \{z \in C : |z| = 1\}$ of the complex numbers of the unit length with multiplication. The convolutive Hilbert algebra $L^2(G)$ can be presented as a sum $\oplus_\alpha \Lambda_\alpha$ of topologically simple algebras, namely, such that have no two-sided closed nontrivial ideals. Since $G$ is a compact group, for each simple algebra $\Lambda_\alpha$ there exists $n(\alpha)$-dimensional matrix algebra $M_{n(\alpha)}(C)$ isomorphic to $\Lambda_\alpha$, so the algebra $L^2(G)$ can be presented in the form of the sum of finite-dimensional matrix algebras.

The first Peter-Weyl theorem states that Hilbert algebra $L^2(G)$ can be considered as a closure of the Hilbert sum of topological simple algebras that is isomorphic to the matrix algebras $M_{n(\alpha)}(C)$ as subspaces $L^2(G)$. Each algebra $\Lambda_\alpha$ consists of elements that are continuous functions on the group $G$ .

The second Peter-Weyl theorem establishes: let $G$ be a compact group, $H$ be a separable Hilbert space, and let $U : G \to U(H)$ be a unitary representation group $G$ in $H$ then the separable Hilbert space $H$ can be represented as a Hilbert sum of finite-dimensional irreducible representations.

There is extensive literature on harmonic analysis and the Fourier theory as its special case, the revision of which is beyond the scope of the present paper [1-7, 15-21].

Applying the results of the Peter-Weyl theorems, we generalize the definitions of the Fourier transforms and study their basic properties. The Fourier transform $F$ of a function of $L^2(G)$ is a function defined on the set of unitary representations of the group $G$ by the equality

$$F(\psi)(\alpha) = \int \psi(g) M_\alpha(g^{-1})\, d\mu(g)$$

where $\mu$ is a normalized Haar measure on the group $G$ such that $\mu(G) = 1$. So, let $\mathrm{M}(G)$ be the space of Haar measures defined on the $\sigma$ -algebra generated by open subsets of $G$ then we can define the Fourier transform $F : \mathrm{M}(G) \to BC(G)$ by

$$F(\mu)(\alpha) = \int M_\alpha(g^{-1})\, d\mu(g),$$

where $BC(G)$ is a set of all bounded and continuous functions on $G$ .

We define the Stieltjes algebra $\hat{\mathrm{M}}(G)$ as $\hat{\mathrm{M}}(G)=\{F(\mu)\ :\ \mu\in\mathrm{M}(G)\}$. Let $T:G\to LB(H,H)$ be a representation of $G$ in the space $LB(H,H)$ of bounded linear operators on the Hilbert space $H$ and let $T$ be a bounded $C_0$-semigroup on $H$. The generalized Fourier-Stieltjes calculus $\Phi$ is defined as the morphism $\Phi:\hat{\mathrm{M}}(G)\to LB(H,H)$ so that

$$\Phi_T(F(\mu))=\int T(g)\,d\mu(g).$$

Generalized Fourier-Stieltjes calculus $\Phi$ satisfies the following norm condition

$$\left\|\Phi_T(F(\mu))\right\|\le\sup_{g\in G}\|T(g)\|\|F(\mu)\|$$

for all $F(\mu)\in\hat{\mathrm{M}}(G)$.

## 2 Some of the Haar measure properties

Let $H$ be a Hilbert space with the scalar product $(,)$. Let $LB(H,H)$ be a space of all linear bounded operators from $H$ to $H$, and let $C(H,H)$ be a space of all compact operators.

Let us denote a compact separable topological group by $G$ and a compact subgroup of $G$ by $\tilde{G}$. There exist the projection $\pi$ from $G$ on $\hat{G}=G/\tilde{G}$, if $\mu$ is a Haar measure on $G$ then $\hat{\mu}=\mu\pi^{-1}$ is a Haar measure on the subgroup $\hat{G}=G/\tilde{G}$. Let $\nu$ be a probabilistic measure on $\tilde{G}$.

**Lemma 1.** *Let $f$ be a nonnegative continuous function on $G$ and let*

$$\varphi(g)=\int_{\tilde{G}} f(g\lambda)\,d\nu(\lambda)$$

*then $\varphi$ is a nonnegative continuous function on $\hat{G}=G/\tilde{G}$ and there exists a unique continuous function $\hat{\varphi}$ such that $\varphi=\hat{\varphi}\pi$.*

Proof. Continuity of $\varphi$ easily follows from continuity of $f$. To show uniqueness, we assume $\pi(g_1)=\pi(g_2)$ so $g_1^{-1}g_2\in\tilde{G}$ and we have

$$\varphi(g_1)=\int_{\tilde{G}} f(g_1\lambda)\,d\nu(\lambda)=$$

$$=\int_{\tilde{G}} f\left(g_1\left(g_1^{-1}g_2\lambda\right)\right)d\nu(\lambda)=\varphi(g_2).$$

For all $\hat{g}=\pi(g)$, the function $\hat{\varphi}$ is uniquely defined on $\hat{G}=G/\tilde{G}$ so that $\varphi=\hat{\varphi}\pi$. The uniqueness of $\hat{\varphi}$ follows from the surjectivity of mapping $\pi$.

**Lemma 2.** ***Let $E$ be a Borel set in $G$ and let there exist a uniquely defined measurable function $\hat{\varphi}_E$ on $\hat{G}=G/\tilde{G}$ such that***

$$\hat{\varphi}_E(\pi(g))=\nu\left(g^{-1}E\cap\tilde{G}\right)=\varphi_E(g)$$

***for any $g\in G$, then the equality***

$$\int\hat{\varphi}_E(g)\,d\hat{\mu}(g)=\mu(E)$$

***holds for all Borel sets in $E$.***

Proof. Assume $g_0\in G$, we have

$$\int\varphi_{g_0E}(g)\,d\mu(g)=\int\nu\left(g^{-1}g_0E\cap\tilde{G}\right)d\mu(g)=$$

$$=\int\nu\left(\left(g^{-1}g_0\right)^{-1}E\cap\tilde{G}\right)d\mu(g)=$$

$$=\int\varphi_E\left(g^{-1}g_0\right)d\mu(g)=\int\varphi_E(g)\,d\mu(g),$$

so, we have that $\int\hat{\varphi}_E(g)\,d\hat{\mu}(g)$ defines a left-invariant measure on Borel sets. Let $E_C$ be a compact set then $\int\hat{\varphi}_E(g)\,d\hat{\mu}(g)=\mu(E_C)$ when $E_C$ is a union of cosets of $\tilde{G}$. We have that there exists $\varphi_E=\hat{\varphi}_E\pi$ when for each Borel set $E$, $\varphi_E(g)=\nu\left(g^{-1}E\cap\tilde{G}\right)$, so, we obtain the statement of the lemma.

## 3 Representation of the compact group

**Definition 1.** *Let $H$ be a Hilbert space and $G$ be a compact group, a group structural preserving endomorphism $\rho:G\times H\to H$ is called a linear representation of the group $G$ on a Hilbert space $H$.*

Let the compact group $G$ be equipped with a Haar measure $\mu$.

Let $U(H)$ be the group of unitary operators on the Hilbert space $H$ and let $L(H,H)$ be a $C^*$-algebra of all continuous linear mappings $H\to H$. Since for any unitary representation $U_1:L^1(G)\to L(H,H)$, the inequality $\|U_1(\varphi)\|\le\|\varphi\|_1$ holds for all $\varphi\in L^1(G)$, the representation $U_1:L^1(G)\to L(H,H)$ is a continuous mapping.

Let $\mathrm{M}^1(G)$ be a space of all regular Borel measures, $\mathrm{M}^1(G)$ is an unital involution Banach algebra. Since each integrable function $\varphi \in L^1(G)$ corresponds to the regular Borel measure $\varphi d\mu$ we have the embedding $M^1(G) \Rightarrow L^1(G)$.

Now let $U: G \to U(H)$ be a unitary representation of $G$ in a separable Hilbert space $H$, we define a weak calculus $\Phi: M^1(G) \to L(H,H)$ given by

$$\Phi_\mu = \int \langle U(g)\psi, \varphi\rangle \; d\mu(g)$$

for all $\psi, \varphi \in H$ and where $\mu$ is a regular Borel measure. Since $U(g)$ is a unitary we have $\|U(g)(\psi)\| = \|\psi\|$, the mapping $g \mapsto \langle U(g)\psi, \varphi\rangle$ is continuous and bounded. Straightforward considerations yield the estimation

$$|\Phi_\mu| \le \|\mu\| \|\psi\| \|\varphi\|,$$

which guarantees the boundness of the linear form $\Phi_\mu(\psi)$.

**Statement (**Riesz representation**) 1.** ***For each continuous linear functional*** $\psi^* \in H^*$ ***there exists one and only one vector*** $\tilde{\psi} \in H$ ***such that***

$$\psi^*(x) = (\psi, \tilde{\psi})$$

***holds for all*** $\psi \in H$**.**

Combining the definition of the weak calculus $\Phi: M^1(G) \to L(H,H)$ and Reisz representation theorem, we obtain that there exists a unique element $\breve{U}(\mu)(\psi)$ of the Hilbert space $H$ such that equality

$$\Phi_\mu = \langle \breve{U}(\mu)(\psi), \varphi\rangle = \int \langle U(g)(\psi), \varphi\rangle \; d\mu(g)$$

holds for all $\psi, \varphi \in H$. So, we have the following weak equality

$$\breve{U}(\mu)(\psi) = \int U(g)(\psi) \; d\mu(g).$$

**Lemma 3.** ***Let*** $U: G \to U(H)$ ***be a unitary representation, then the mapping*** $\breve{U}: M^1(G) \to L(H,H)$ ***is a unitary representation*** $M^1(G)$ ***in*** $L(H,H)$***. The restriction*** $U_{st}: L^1(G) \to L(H,H)$ ***of*** $\breve{U}$ ***on*** $L^1(G)$ ***is nondegenerate.***

Proof. Let $\psi, \varphi \in H$ we write

$$\begin{aligned}
&\langle \breve{U}(\mu * \nu)(\psi), \varphi\rangle = \\
&= \int \langle U(g)\psi, \varphi\rangle \, d(\mu * \nu)(g) = \\
&= \int\int \langle U(gh)\psi, \varphi\rangle \, d\mu(g) d\nu(h) = \\
&= \int\int \langle U(g)\psi, U^*(h)\varphi\rangle \, d\mu(g) d\nu(h) = \\
&= \int \langle \breve{U}(\nu)\psi, U^*(h)\varphi\rangle \, d\mu(h) = \\
&= \langle \breve{U}(\mu)\breve{U}(\nu)\psi, \varphi\rangle,
\end{aligned}$$

so, we have $\breve{U}(\mu * \nu) = \breve{U}(\mu)\breve{U}(\nu)$.

From the unitarity of $U: G \to U(H)$ follows $(U(g))^* = U(g^{-1})$ and we obtain $(\breve{U}(\mu(g)))^* = \breve{U}(\overline{\mu}(g^{-1}))$.

The representation $\breve{U}$ is nondegenerate if and only if $\breve{U}(\phi d\mu(g))\psi = 0$ for all $\phi \in L^1(G)$ yields $\psi = 0$. Let the system $\{W_\varepsilon\}$ be a neighborhood base of the identity element of the group $G$ such that $W_{\varepsilon_2} \subseteq W_{\varepsilon_1}$ of all $\varepsilon_2 \prec \varepsilon_1$. For each $\varepsilon$, there is a strictly positive function $\phi_\varepsilon: G \to R$ with a compact support contained in $gW_\varepsilon$ and such that $\int \phi_\varepsilon \, d\mu(g) = 1$.

For any $\psi \in H$ and any $\theta > 0$, there is some $\varepsilon$ such that

$$\|U(\tilde{g})(\psi) - U(g)(\psi)\| \le \theta$$

for all $\tilde{g} \in gW_\varepsilon$. For all $\varphi \in H$, we have

$$\begin{aligned}
&\langle \breve{U}(\phi_\varepsilon \, d\mu)(\psi) - U(g)(\psi), \varphi\rangle = \\
&= \int \langle U(\tilde{g})(\psi) - U(g)(\psi), \varphi\rangle \phi_\varepsilon(\tilde{g}) \, d\mu(\tilde{g}),
\end{aligned}$$

so, we obtain

$$\begin{aligned}
&\|\breve{U}(\phi_\varepsilon \, d\mu)(\psi) - U(g)(\psi)\|^2 = \\
&= \int \left\langle \begin{matrix} U(\tilde{g})(\psi) - U(g)(\psi), \\ \breve{U}(\phi_\varepsilon \, d\mu)(\psi) - U(g)(\psi) \end{matrix} \right\rangle \phi_\varepsilon(\tilde{g}) \, d\mu(\tilde{g}) \le \\
&\le \int \begin{matrix} \|U(\tilde{g})(\psi) - U(g)(\psi)\| \times \\ \|\breve{U}(\phi_\varepsilon \, d\mu)(\psi) - U(g)(\psi)\| \phi_\varepsilon(\tilde{g}) \, d\mu(\tilde{g}) \end{matrix} \le \\
&\le \|\breve{U}(\phi_\varepsilon \, d\mu)(\psi) - U(g)(\psi)\| \theta,
\end{aligned}$$

so, $\breve{U}\left(\phi d\mu\left(g\right)\right)\psi = 0$ holds for $\phi = \phi_{\varepsilon}$ and $g = e$, $U\left(e\right) = I$ only if $\psi = 0$, thus the restriction of $\breve{U}$ to $L^{1}\left(G\right)$ is nondegenerate.

**Lemma 4**. ***Let*** $U_{1} : L^{1}\left(G\right) \to L\left(H,H\right)$ ***be a nondegenerate unitary representation of*** $L^{1}\left(G\right)$ ***in*** $L\left(H,H\right)$ ***then there exists a unique representation*** $U : G \to U\left(H\right)$ ***of the group*** $G$ ***in the group*** $U\left(H\right)$ ***of unitary operators on the Hilbert space*** $H$ ***such that*** $U_{1} = U_{st}$ ***.***

Proof. Assume $U_{1} : L^{1}\left(G\right) \to L\left(H,H\right)$ is a nondegenerate unitary representation. The closure of the span by set $\left\{U_{1}\left(\varphi\right)\psi : \quad \varphi \in L^{1}\left(G\right), \quad \psi \in H\right\}$ coincides with whole Hilbert space $H$. Let us for any taking $g \in G$ determine a set $\left\{W_{\varepsilon}\right\}$ of neighborhoods $W_{\varepsilon}$ of the identity element of $G$ and the set $\left\{\phi_{\varepsilon}\right\}$ of functions $\phi_{\varepsilon} : G \to R$ with compact support contained in $gW_{\varepsilon}$ and such that $\int \phi_{\varepsilon}\, d\mu\left(g\right) = 1$. Let $\delta$ be a delta-function then we obtain

$$\left\|\phi_{\varepsilon} * \varphi - \delta\left(g\right) * \varphi\right\|_{1} = $$
$$= \left\|\delta\left(g\right) * \delta\left(g^{-1}\right) * \phi_{\varepsilon} * \varphi - \delta\left(g\right) * \varphi\right\|_{1} \le$$
$$\le \left\|\delta\left(g^{-1}\right) * \phi_{\varepsilon} * \varphi - \varphi\right\|_{1} \underset{\varepsilon \to 0}{\to} 0.$$

We have

$$\left\| \begin{matrix} U_{1}\left(\phi_{\varepsilon}\right)\left(\sum_{k} U_{1}\left(\varphi_{k}\right)\psi_{k}\right) - \\ -\sum_{k} U_{1}\left(\delta\left(g\right) * \varphi_{k}\right)\psi_{k} \end{matrix} \right\| \le$$
$$\le \sum_{k} \left\| \begin{matrix} U_{1}\left(\phi_{\varepsilon}\right) \cdot U_{1}\left(\varphi_{k}\right) - \\ -U_{1}\left(\delta\left(g\right) * \varphi_{k}\right) \end{matrix} \right\| \left\|\psi_{k}\right\| \xrightarrow[\varepsilon \to 0]{} 0.$$

Let $\gamma = \sum_{k} U_{1}\left(\varphi_{k}\right)\left(\psi_{k}\right)$, we can write

$$U\left(g\right)\left(\gamma\right) = U\left(g\right)\left(\sum_{k} U_{1}\left(\varphi_{k}\right)\left(\psi_{k}\right)\right) =$$
$$= \sum_{k} U_{1}\left(\delta\left(g\right) * \varphi_{k}\right)\psi_{k}$$

thus we obtain morphism $U\left(g\right)$ from the spanned $\left\{U_{1}\left(\varphi\right)\psi : \quad \varphi \in L^{1}\left(G\right), \quad \psi \in H\right\}$ to $H$ thus $U\left(g\right)$ maps spanned $\left\{U_{1}\left(\varphi\right)\psi : \quad \varphi \in L^{1}\left(G\right), \quad \psi \in H\right\}$ to itself, so we have

$$\left\|U_{1}\left(\varphi_{k}\right)\right\| \le \left\|\varphi_{k}\right\|_{1} = 1$$

thus for all $g \in G$, we have $\left\|U\left(g\right)\left(\gamma\right)\right\| \le \left\|\gamma\right\|$. So, for all $g \in G$ we have $\left\|U\left(g\right)\right\| \le 1$.

For all $g, \tilde{g} \in G$ and all $\varphi \in L^{1}\left(G\right)$, we write

$$U\left(g\tilde{g}\right) \cdot U_{1}\left(\varphi\right) = U_{1}\left(\delta\left(g\tilde{g}\right) * \varphi\right) =$$
$$= U_{1}\left(\delta\left(g\right) * \left(\delta\left(\tilde{g}\right) * \varphi\right)\right) =$$
$$= U\left(g\right) \cdot U_{1}\left(\delta\left(g\right) * \varphi\right) =$$
$$= U\left(g\right) \cdot U\left(\tilde{g}\right) \cdot U_{1}\left(\varphi\right),$$

so, we have that $U$ maps group identity element to the identity element of the Hilbert space and $U\left(g\tilde{g}\right) = U\left(g\right) \cdot U\left(\tilde{g}\right)$ in the Hilbert space.

Mapping $U$ is a continuous isometry since the equality

$$\left\|\psi\right\| = \left\|U\left(g^{-1}g\right)\left(\psi\right)\right\| =$$
$$= \left\|U\left(g^{-1}\right)\left(U\left(g\right)\left(\psi\right)\right)\right\| = \left\|U\left(g\right)\left(\psi\right)\right\| = \left\|\psi\right\|$$

holds for all $g \in G$ and all $\psi \in H$.

Since the dual to the space $L^{1}$ is the space isomorphic to $L^{\infty}$, we have

$$\int \left(\varphi * \zeta\right)\left(g\right) f\left(g\right) d\mu\left(g\right) =$$
$$= \int \varphi\left(\tilde{g}\right)\left(\int \left(\delta\left(\tilde{g}\right) * \zeta\right)\left(g\right) f\left(g\right) d\mu\left(g\right)\right) d\mu\left(\tilde{g}\right)$$

for all $\varphi, \zeta \in L^{1}\left(G\right)$ and all $f \in L^{\infty}\left(G\right)$.

Let $\psi, \tilde{\psi} \in H$ then the linear form $\varphi \mapsto \left\langle U_{1}\left(\varphi\right)\left(\psi\right), \tilde{\psi}\right\rangle$ is continuous on $L^{1}\left(G\right)$ thus there exists a function $f \in L^{\infty}\left(G\right)$ such that

$$\left\langle U_{1}\left(\varphi\right)\left(U_{1}\left(\zeta\right)\left(\psi\right)\right), \tilde{\psi}\right\rangle = \left\langle U_{1}\left(\varphi * \zeta\right)\left(\psi\right), \tilde{\psi}\right\rangle =$$
$$= \int \varphi\left(g\right)\left\langle U_{1}\left(\delta\left(g\right) * \zeta\right)\left(\psi\right), \tilde{\psi}\right\rangle d\mu\left(g\right) =$$
$$= \int \left\langle U\left(g\right)U_{1}\left(\zeta\right)\left(\psi\right), \tilde{\psi}\right\rangle \varphi\left(g\right) d\mu\left(g\right) =$$
$$= \left\langle U_{1}\left(\varphi\right)U_{1}\left(\zeta\right)\left(\psi\right), \tilde{\psi}\right\rangle$$

holds for all $\psi, \tilde{\psi} \in H$. So, since the span by set $\left\{U_{1}\left(\varphi\right)\psi : \quad \varphi \in L^{1}\left(G\right), \quad \psi \in H\right\}$ is dense in $H$, we have obtained $U_{1}\left(\varphi\right) = U_{st}\left(\varphi\right)$.

As a corollary of the lemmata 3 and 4, we obtain an important theorem.

**Theorem 1**. ***Let*** $U:G\to U(H)$ ***be a unitary representation of the group*** $G$ ***and*** $U_{st}:L^1(G)\to L(H,H)$ ***be a unitary representation of*** $L^1(G)$ ***are defined as in the lemmata 3, 4. Then there exists a bijective mapping*** $\Upsilon:U\to U_{st}$ ***between the set of unitary representations of the group*** $G$ ***and the set of nondegenerate topologically irreducible unitary representations of the Banach algebra*** $L^1(G)$.

So, we consider a unitary representation $U:G\to U(H)$ of the group $G$ and construct a unitary representation $\breve{U}:\mathrm{M}^1(G)\to L(H,H)$ of the unitary Banach algebra $\mathrm{M}^1(G)$ then we restrict $\breve{U}:\mathrm{M}^1(G)\to L(H,H)$ to Banach algebra $L^1(G)$ (this restriction is denoted by $U_{st}(\varphi)$). Next, we consider a nondegenerate unitary representation $U_1:L^1(G)\to L(H,H)$ and show that $U_1(\delta(g)*\varphi_k)\psi$ tends to $U(g)(\psi)$ for all $g\in G$ and $\psi\in H$.

# 4 The Generalized Fourier transform

Let $G$ be a compact Hausdorff group equipped with a Haar measure $\mu$.

A complete Hilbert algebra of the square-integrable functions on the group $G$ is denoted by $L^2(G)$. By the Peter-Weyl theorem, algebra $L^2(G)$ can be presented as an orthogonal sum $\oplus_{\alpha\in R}\Lambda_\alpha=L^2(G)$ of topologically simple algebras $\Lambda_\alpha$, where $\Lambda_\alpha$ equals to matrix algebra $M_{n(\alpha)}(C)$ of dimension $(n(\alpha))^2$, where $\alpha$ is a finite-dimensional representation. Each function $\Lambda_\alpha:G\to M_{n(\alpha)}(C)$ is continuous on the compact group $G$.

**Definition 2**. *The set of all equivalence classes of an irreducible representation of the group* $G$ *is called* $\hat{G}$.

**Theorem** ***(first Peter-Weyl) 2. The separable Hilbert algebra*** $L^2(G)$ ***can be presented as an orthogonal sum*** $\oplus_{\alpha\in R}\Lambda_\alpha$***, where each simple topological algebra*** $\Lambda_\alpha$ ***is isomorphic to a matrix algebra*** $M_{n(\alpha)}(C)$ ***of dimension*** $(n(\alpha))^2$***. The unit element of*** $\Lambda_\alpha$ ***is a continuous function*** $\phi_\alpha$ ***that satisfies the condition*** $\phi_\alpha(g)=\overline{\phi_\alpha(g^{-1})}$ ***for all*** $g\in G$***. For each function*** $\psi\in L^2(G)$***, there exists a representation*** $\psi=\sum_\alpha^{`}\psi*\phi_\alpha$.

The presentation $\psi=\sum_\alpha^{`}\psi*\phi_\alpha$ follows from $\phi_\alpha=\sum_{k=1,..,n(\alpha)}^{`}e_k$ and so that $\sum_{k=1,..,n(\alpha)}^{`}\psi*e_k=\psi*\phi_\alpha$.

**Theorem** ***(second Peter-Weyl) 3. Let*** $U:G\to U(H)$ ***be a unitary representation of the compact Hausdorff group in a separable Hilbert space*** $H$***. Then, first, for each finite-dimensional representation*** $\alpha$***, the mapping*** $U_{st}(\phi_\alpha)$ ***is an orthogonal projection*** $H\mapsto E(\alpha)$***; second, each*** $E(\alpha)\neq\{0\}$ ***is invariant relative to*** $U$ ***and restriction*** ${U_{st}}^\alpha$ ***of*** $U$ ***to*** $E(\alpha)$ ***can be represented as*** ${U_{st}}^\alpha=\oplus_{\alpha\in R}M_{\bar{\alpha}}$.

Each element $\Lambda_\alpha$ uniquely corresponds with a continuous function, so that for each finite-dimensional representation $\alpha$ there is a decomposition $\Lambda_\alpha=\oplus_{1\le k\le n(\alpha)}\Lambda_\alpha*m_k$ where $m_k$ is an irreducible idempotent, and so that $\phi_\alpha=\sum_{k=1,..,n(\alpha)}^{`}m_k$. Let $\{a_k\}_{1\le k\le n(\alpha)}$ be a Hilbert basis in $\Lambda_\alpha*m_1$ with the condition $a_k\in m_k*\Lambda_\alpha*m_1$.

**Definition 3**. *For every finite-dimensional representation* $\alpha$*, we define a matrix* $M_\alpha(g)$ *of* $n(\alpha)\times n(\alpha)$*-dimension with coefficients*

$$a_{ij}(g)=(n(\alpha))^{-1}\left(a_i(g)*\overline{a_j(g^{-1})}\right)$$

*for* $1\le i\le n(\alpha)$ *and* $1\le j\le n(\alpha)$.

From the definition we have $a_{ii}=m_i$.

Applying the Peter-Weyl theorem we are going to generalize the Fourier transform.

Let $A:G\to U(H)$ be a continuous mapping $g\mapsto A(g)(\psi)$ given by

$$A(g)=M_\alpha(g^{-1}h)$$

for any element $\psi \in L^1(G)$ of the separable Hilbert space $H$.

Applying our notations, for any $\psi \in L^2(G)$, we have

$$\psi = \sum_{\alpha} \psi * \phi_\alpha = \sum_{\alpha} \sum_{k=1,..,n(\alpha)} \psi * m_{kk}(\alpha),$$

next, we write

$$\sum_{k=1,..,n(\alpha)} (\psi * m_{kk}(\alpha))(g) =$$
$$= \sum_{k=1,..,n(\alpha)} \int \psi(h)(m_{kk}(\alpha))(h^{-1}g)\, d\mu(h) =$$
$$= n(\alpha) tr\left(\int \psi(h) M_\alpha(h^{-1}g)\, d\mu(h)\right).$$

**Definition 4**. *The Fourier transform $F(\psi)$ of the function $\psi \in L^1(G)$ is a mapping defined by*

$$F(\psi)(\alpha) = \int \psi(h) M_\alpha(h^{-1})\, d\mu(h).$$

So, the Fourier transform is a mapping defined in the domain of irreducible representations of the compact group $G$. The morphism $F(\psi)(\alpha)$ is automorphism $C^{n(\alpha)} \mapsto C^{n(\alpha)}$ that is represented by matrices.

Example. If we assume $G = S^1$ a circle group then $\hat{G} = \hat{S}^1 = Z$ and $F(\psi)(m) = \hat{\psi}(m)$ are classical Fourier coefficients of the function $\psi \in L^1(S^1)$.

For all $\psi \in L^2(G)$, $h \in G$, the inverse Fourier transform can be defined by

$$\psi(h) = \sum_{\alpha} n(\alpha) tr(F(\psi)(\alpha) M_\alpha(h)).$$

**Definition 5**. *The set $\bigcap_\alpha M_{n(\alpha)}(C)$ is denoted by $\Theta(\hat{G})$.*

*Let $A$ be a complex matrix of $n \times n$ dimension, let $\{\lambda_k\}_{1,...,n}$ be the set of nonnegative square roots of eigenvalues of $A^*A$, the Heumann norm $\|A\|_{\varphi(p)}$ of the matrix $A$ is defined by*

$$\|A\|_{\varphi(p)} = \left(\sum_{k=1,...,n} \lambda_k^p\right)^{\frac{1}{p}} \quad \textit{for} \quad 1 \le p < \infty \quad \textit{and}$$

$$\|A\|_{\varphi(\infty)} = \max_{k=1,...,n} \lambda_k.$$

**Definitions 6**. *The space $L^p(\hat{G})$ is defined as*

$$L^p(\hat{G}) = \left\{ f \in \bigcap_\alpha M_{n(\alpha)}(C) \;:\; \|f\|_p < \infty \right\},$$

*where the norm $\|f\|_p$ is defined as*

$$\|f\|_p = \langle f, \tilde{f} \rangle = \left(\sum_\alpha \phi_\alpha \|f(\alpha)\|^p_{\varphi(p)}\right)^{\frac{1}{p}} \quad \textit{for any}$$

*sequence $\{\phi_\alpha\}$ of the real numbers such that $\phi_\alpha \ge 1$.*

**Lemma 5.** For each set $\{\phi_\alpha\}$ of real numbers larger than one, the space $L^p(\hat{G})$ is a Banach space.

For any fixed $p \in [1, \infty]$, we take $q = \dfrac{p}{p-1}$ so that for all $f \in L^p(\hat{G})$ and all $\tilde{f} \in L^q(\hat{G})$ then there is a well-defined value $\langle f, \tilde{f} \rangle = \sum_\alpha \phi_\alpha tr(\tilde{f}_\alpha^*, f_\alpha)$ such that

$$\langle \tilde{f}, f \rangle = \overline{\langle f, \tilde{f} \rangle}$$

and the Holder inequality holds in the form

$$\left|\langle f, \tilde{f} \rangle\right| \le \|f\|_p \|\tilde{f}\|_q.$$

In a special case $p = 2$, we have the following lemma.

**Lemma 6**. For each set $\{\phi_\alpha\}$ of real numbers larger than one, the space $L^2(\hat{G})$ is a Hilbert space. The inner product is defined by $\langle f, \tilde{f} \rangle = \sum_\alpha \phi_\alpha tr(\tilde{f}_\alpha^*, f_\alpha)$ and so that $\langle f, f \rangle = \|f\|_2^2$.

**Theorem 4.** ***Let $\{\phi_\alpha\}$ be a fixed set of real numbers larger than one. Then for any $f \in L^p(\hat{G})$ and $\tilde{f} \in L^q(\hat{G})$ such that $\frac{1}{p} + \frac{1}{q} = 1$, we claim: first, $f \cdot \tilde{f} \in L^1(\hat{G})$ so that $\|f \cdot \tilde{f}\|_1 \le \|f\|_p \|\tilde{f}\|_q$; second, if $1 \le p \le q \le \infty$ then $L^p(\hat{G}) \subseteq L^q(\hat{G})$ so that if $f \in L^p(\hat{G})$ then $\|f\|_q \le \|f\|_p$; third, $\|f \cdot \tilde{f}\|_p \le \|f\|_p \|\tilde{f}\|_p$ for all $f \cdot \in L^p(\hat{G})$ and $\tilde{f} \in L^p(\hat{G})$ so that $f \cdot \tilde{f} \in L^p(\hat{G})$.***

The proof is straightforward and its schema can be found in E. Hewitt and K. Ross [13].

Thus, we formulate the results of the article in the two theorems which are generalizations of the Fourier theory.

**Theorem** *(first theorem) 5.* ***Let*** $G$ ***be a compact group then the mapping*** $F: L^2(G) \to L^2(\hat{G})$ ***defined by***

$$F(\psi)(\alpha) = \int \psi(g) M_\alpha(g^{-1}) d\mu(g)$$

***is an isometric isomorphism.***

***For each element*** $\psi \in L^2(G)$***, we have a representation***

$$\psi = \sum_\alpha n(\alpha) \sum_{i,k=1,\dots,n(\alpha)} \langle\langle F(\psi)(\alpha)(e_i(\alpha)), (e_k(\alpha))\rangle\rangle \times \phi_{ik}(\alpha),$$

***where*** $\{e_i(\alpha)\}_{i=1,\dots,n(\alpha)}$ ***is an orthonormal basis in*** $C^{n(\alpha)}$ ***and coordinate functions*** $\phi_{ik}$ ***are defined as***

$$\phi_{ik}(\alpha)(g) = \langle M_\alpha(g) e_i(\alpha), e_k(\alpha)\rangle$$

***for all*** $g \in G$ ***and*** $i, k = 1, \dots, n(\alpha)$.

**Theorem** *(second theorem) 7.* ***Let*** $G$ ***be a compact group then the inverse Fourier transform*** $F^{-1}: L^2(\hat{G}) \to L^2(G)$ ***is defined by***

$$\psi(g) = \sum_\alpha n(\alpha) tr(F(\psi)(\alpha) M_\alpha(g))$$

***for any Fourier transform*** $F(\psi) \in L^2(\hat{G})$ ***of*** $\psi \in L^2(G)$ ***and the series converges in*** $L^2$.

Now, we generalize the Fourier-Stieltijes calculus. Let $T: G \to LB(H,H)$ be a continuous and bounded representation of the group $G$ in the space of all linear bounded operators from the Hilbert space $H$ to $H$. Let $\hat{\mathrm{M}}(G)$ be a Stiejes algebra of all Fourier transforms $F(\mu)$ for all $\mu \in \mathrm{M}(G)$. We define ***the generalized Fourier-Stieltjes calculus*** $\Phi$ ***for*** $T$ ***as a morphism*** $\hat{\mathrm{M}}(G) \mapsto LB(H,H)$ ***given by***

$$\Phi_T(F(\mu)) = \int T(g) d\mu(g).$$

For generalized Fourier-Stieltjes calculus $\Phi$, we have the following norm inequality

$$\|\Phi_T(F(\mu))\| \le \sup_{g \in G} \|T(g)\| \|F(\mu)\|_{\hat{\mathrm{M}}(G)}$$

that holds for all $F(\mu) \in \hat{\mathrm{M}}(G)$.

# 5 The Hille-Phillips calculus

We are going to consider a special case of $T: G \to LB(H,H)$ when a continuous and bounded representation $T = T_t$ where $\{T_t, \ t \ge 0\}$ is a bounded $C_0$ - semigroup on a separable Hilbert space $H$. The Fourier-Stieltjes calculus $\Phi: \hat{\mathrm{M}}(R_+) \to LB(H,H)$ for the semigroup is given by

$$\Phi_T(F(\mu)) = \int_{[0,+\infty)} T_t d\mu(t).$$

The Laplace transform $Lap(\mu): \overline{C_+} \to C$ of the measure $\mu \in \mathrm{M}(R_+)$ is defined by

$$Lap(\mu)(z) = \int_{[0,+\infty)} \exp(-zt) d\mu(t)$$

for $z \in C$.

We define the set $LAP(C_+)$ as $LAP(C_+) = \{Lap(\mu): \mu \in \mathrm{M}(R_+)\}$ and equip it with its natural norm $\|Lap(\mu)\|_{LAP} = \|\mu\|_{\mathrm{M}}$ for all $\mu \in \mathrm{M}(R_+)$.

**Definition 7.** ***The isomorphism*** $\Psi_T$ ***from*** $Lap(C_+)$ ***to*** $LB(H,H)$ ***given by***

$$\Psi_T(Lap(\mu)) = \int_{[0,+\infty)} T_t d\mu(t)$$

***is called a Hille-Phillips calculus for bounded*** $C_0$ ***- semigroup*** $\{T_t, \ t \ge 0\}$.

For all $\mu \in \mathrm{M}(R_+)$ the Hille-Phillips calculus satisfies the inequality

$$\|\Psi_T(Lap(\mu))\| \le \sup_{t \in R_+} \|T_t\| \|\mu\|_{\mathrm{M}}.$$

The straightforward calculation yields the next theorem.

**Theorem (Hille-Phillips) 8.** ***Let*** $\Psi_T$ ***be a Hille-Phillips calculus for a bounded*** $C_0$ ***- semigroup*** $\{T_t, \ t \ge 0\}$ ***on a separable Hilbert space*** $H$***. Then there exists a generator*** $-A: H \to H$ ***such that*** $\Psi_T((\lambda + z)^{-1}) = (\lambda + A)^{-1}$ ***for all*** $z \in C_+$***. The Hille-Phillips calculus is uniquely defined by its generator as follows*** $\Psi_T(Lap(\delta(t))) = \exp(-tA)$ ***for all*** $t \ge 0$ ***where*** $\delta$ ***notes a delta-function.***

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