Bond pricing under sticky OU process

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Abstract: In this paper, we derive the conditional characteristic function of the sticky Ornstein-Uhlenbeck (OU) process and explore bond pricing under this framework. We systematically transform the standard OU process into the sticky OU process by incorporating the time-varying symmetric local time, thereby establishing the existence of a unique weak solution for this modified process. Subsequently, leveraging the infinitesimal generator and its domain, we meticulously compute the conditional characteristic function of the sticky OU process. Following a similar analytical approach, we incorporate the sharpe ratio into our bond pricing methodology, ensuring coherence and rigor in our calculations. Notably, all our findings are presented in closed-form expressions, facilitating straightforward interpretation and application in financial modeling and analysis. This comprehensive treatment not only advances the theoretical understanding of the sticky OU process but also offers practical insights into bond valuation dynamics under this intricate process.

Key-Words: Sticky OU process, Conditional characteristic function, Bond pricing, Closed-form solution

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1 Introduction

The Ornstein-Uhlenbeck (OU) process refers to the state variable X_t satisfying the following stochastic differential equation:

$$dX_t = \kappa(\alpha - X_t)dt + \sigma dW_t,$$

where κ, α and σ are constants. In financial literature, α is often referred to as the long-term mean of a stochastic process X_t. When $\kappa > 0$, X_t is a stochastic process with the property of mean regression. Because if $X_t > \alpha$, X_t is above its long-term average, then $\kappa(\alpha - X_t) < 0$, which X_t moves towards its long-term average α; if $X_t < \alpha$, X_t is below its long-term average, then $\kappa(\alpha - X_t) > 0$, which X_t moves towards its long-term average α.

The essence of the OU process was proposed by [1], in his famous paper on Brownian motion. The authors in [2], initially introduced this process as a model to describe the velocity changes of Brownian particles under the influence of friction. The Ornstein-Uhlenbeck (OU) process, as a pivotal stochastic process model, has consistently garnered significant attention for its rich mathematical properties and diverse statistical analysis methodologies. Its theoretical foundations hold immense academic value and practical relevance, underscoring its enduring appeal and fostering extensive research endeavors among scholars globally. Extensive research efforts, both domestically and internationally, have delved into various aspects of the OU process, including but not limited to parameter estimation, model validation, and comparative analyses with alternative stochastic process models. These investigations have not only deepened our understanding of the OU process's intricacies but also widened its application scope across various disciplines.

The study, [3], adequately studied the simulation and estimation of this process. The study, [4], discussed robust estimation methods for OU processes that protect against outliers and deviations from ideal laws, proposed basic M-estimates and optimal asymptotic linear (AL) estimates. The study, [5], studied a numerical simulation algorithm that is accurately applicable to any time step for the OU process $X(t)$ and its time integration $Y(t)$. The study, [6], provided a fractional order OU process through the Lamperti transformation of fractional Brownian motion. The study, [7], studied and discussed parameter estimation for fractional order OU processes. The study, [8], studied the properties of the reflective OU process and demonstrated that the reflected OU process exhibits both steady-state and transient behaviors that are reasonably manageable. The study, [9], studied the asymptotic behavior of the first passage time probability density function (p.d.f.) of the OU process through a constant boundary under large boundary conditions. The study, [10], investigated the solutions of first-passage time equations for both Wiener process and OU process, specifically considering the general case of time-dependent threshold functions. The study, [11], studied in detail the asymptotic properties of the first passage time p.d.f and its moments of the OU process under an unconstrained condition and a constant boundary condition, and obtained an explicit expression for any order passage time moment that is particularly suitable for calculation. The study, [12], studied the maximum inequality of the OU process. The study, [13], studied the probability density of OU processes with jumps. The study, [14], derived three different expressions for the first hitting time density of the OU process at a predetermined level. The study, [15], studied the main characteristics of the time-varying OU process, such as the covariance function, and proved the formula for the generalized fractional Fokker-Planck equation that describes the one-dimensional probability density function of the analyzed system. For other theoretical and applied research on stochastic process, readers may consult, [16], [17], [18], [19], [20]. These studies not only enrich the theoretical system of OU processes, but also provide a solid theoretical foundation for the application of OU processes in practical problems.

In the realm of finance, the OU process has emerged as a prominent tool for modeling the dynamics of interest rates and other financial asset prices. Distinct from the traditional geometric Brownian motion, the OU process exhibits a distinctive mean-reverting characteristic, which renders it particularly adept at capturing the behavior of certain financial assets. This attribute has garnered widespread interest and attention from scholars both domestically and internationally.

The earliest application of OU process in finance was [21], who used this model to describe the dynamic change of instantaneous interest rate. Therefore it is generally referred to as the Vasicek model in financial literature. The ease of handling and interesting randomness of the Vasicek model in bond pricing make this classic model very popular. The explicit formula for bond prices was originally derived by solving PDE regarding to bond pricing. The study, [22], researched the characteristics of PDE related to bond pricing. They proved that if certain Riccati equations have a solution on a specific maturity date, bond prices will exhibit exponential affine form. In addition, Vasicek's model generates a set of solvable equations due to its specific specifications for drift and volatility, which is consistent with the theoretical descriptions of Duffie and Kan, and is therefore classified as an exponential affine model. Recently, [23], proposed a new approach to address the issues discussed by Duffie and Kan. They pointed out that when the changes in short-term interest rates follow Gaussian dynamics or square root processes, the price of bonds will exhibit an exponential affine function. Unlike before, their technique is to directly determine bond prices through integral linear ordinary differential equations, without the need for Riccati equations. Similarly this method has been successfully applied to solve the bond pricing problem in the Vasicek model. Furthermore, [24], discussed three methods for the closed solution of the Vasicek bond pricing problem, derived this PDE through the martingale method, and determined the bond price through the integral ordinary differential equation. In addition, under the risk-neutral measure commonly employed for valuation and pricing, one could hypothesize a comparable parametrization for an identical process, albeit incorporating an additional parameter in the drift component to model the market price of risk. For instance, refer to [25]. In addition, [26], introduced several different stochastic processes, including but not limited to Vasicek models and exponential Vasicek models, which help to understand the basic characteristics of risk factors describing different asset classes or behaviors. The study, [27], studied the optimal proportional reinsurance and investment strategies in the stock market with the OU process, and derived explicit expressions for the optimal strategy and value function. These expressions are not only applicable to compound Poisson risk model, but also to Brownian motion risk model.

However, although the OU process has shown many advantages in bond pricing, many factors still need to be considered in its practical application. For example, according to surveys, negative interest rate policies are prevalent in developed countries. As shown in Figure 1, the Bank of Japan has adopted a zero interest rate policy in Asian financial markets since February 15, 1999. Furthermore, in response to the economic environment and achieving monetary policy goals, the bank further lowered its benchmark interest rate to -0.1% on January 29, 2016. This negative interest rate level continued until March 19, 2024.

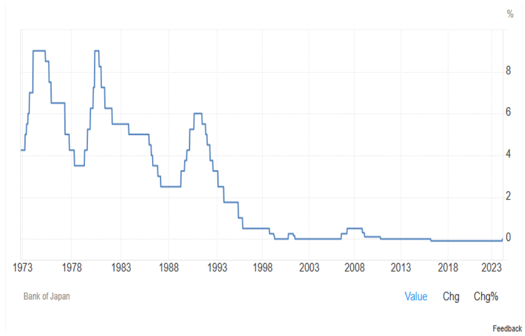

Fig. 1: Japan's benchmark interest rate from 1973 to 2023

Similarly, in order to stimulate economic growth and address the risk of deflation, some countries have adopted corresponding monetary policies. These policies include lowering deposit interest rates and

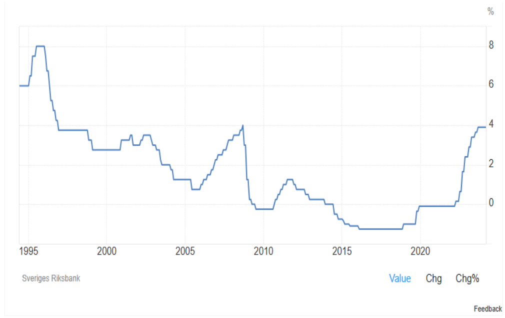

Fig. 2: Overnight deposit rates for Swedish banks from 1995 to 2020

loan benchmark interest rates, and even leading to negative interest rates. As shown in Figure 2, in July 2009, the overnight deposit rate of Swedish banks passively decreased to -0.25%. On February 18, 2015, the interest rate was officially lowered to -1%. It is worth noting that from February 2016 to November 2018, Sweden's overnight deposit interest rate remained at a high negative interest rate level of -1.25%. In addition, the overnight deposit interest rate remained at -0.1% from December 2019 until March 2024. As shown in Figure 3, on December 18, 2014, the Swiss National Bank made a decision to implement a negative interest rate policy of -0.25% on the balance of spot deposit accounts. On January 15, 2015, the Swiss National Bank further lowered the spot deposit rate to -0.75%. It is worth noting that this negative interest rate level (-0.75%) was maintained in the following years until June 15, 2021.

Fig. 3: Swiss bank spot deposit rates from 2001 to 2021

The inflation rate in Switzerland has also shown some fluctuations in the past few decades. Although we cannot directly provide annual inflation rate data that fully corresponds to current deposit interest rates, we can analyze based on general trends. Usually, the inflation rate is influenced by various factors, including monetary policy, economic growth, international commodity prices, etc. When comparing the interest rates of Swiss bank demand deposits with inflation rates, the mutual influence between the two can be observed. When the inflation rate is high, banks may raise deposit interest rates to attract depositors and maintain fund stability; When the inflation rate is low, banks may lower deposit interest rates to reduce costs. However, this relationship is not absolute, as banks' interest rate decisions are also influenced by various other factors. At the same time, the inflation rate in the United States has also shown a similar fluctuating trend. According to data provided by the World Bank, the inflation rate in the United States has also experienced multiple fluctuations over the past two decades. These changes are also influenced by monetary policy, economic growth, and the international economic environment. When comparing the inflation rates of Switzerland and the United States, we can see that there is a certain difference between the two. This difference may stem from factors such as the different economic structures, monetary policies, and external environments of the two countries. However, it should be noted that changes in inflation rates have similar impacts on the economies and financial markets of both countries, which may lead to currency depreciation, price increases, and asset price fluctuations.

Compared to traditional economies, in these negative interest rate markets, interest rates are no longer freely moving in the high interest rate range far from zero, and there is a phenomenon of long-term interest rate maintenance at a certain level. For the commonly occurring clustering phenomenon, we found that the sticky diffusion process can be used to describe this interest rate dynamics. In 1952, [28], discussed the sticky boundary behavior of Markov processes, and further investigated the sticky diffusion process, [29], [30]. In [31], the author studied the stochastic differential equations of Markov processes with sticky points and proved the existence and uniqueness of weak solutions for the process, but pointed out that there is no strong solution. The study, [32], discussed and studied the sticky Brownian motion, emphasizing that the process has weak solutions but no strong solutions, verifying Skorokhod's conjecture. The study, [33], studied the sticky Brownian motion on a state space of $[0, +\infty)$, discussed its boundary behavior, and calculated its infinitesimal generator and steady-state distribution. The authors in [34], studied the sticky diffusion process as a one-dimensional Markov process with spatial delay, and provided a path for the delay process based on SDE and the occupancy formula with symmetric local time. In addition, [35], [36], studied the sticky skew Brownian motion and sticky skew CIR process, and proved the relevant properties.

It is precisely because of in-depth research on the sticky diffusion process that many scholars have found that this model can be applied to the financial field. The unique characteristics of the sticky diffusion process can more flexibly characterize the price clustering phenomenon that occurs in financial markets. The study, [37], [38], provided some research on price clustering phenomena.

After comprehensive thoroughly, we found that the pricing theory under the sticky process is still in its early stages. Currently, only, [39], have conducted preliminary discussions on bond pricing under the sticky Brownian motion. Based on the actual situation of the international interest rate market, we have chosen the sticky OU model as the research object, which not only helps to fill this research gap, but also provides more scientific decision-making basis for practical applications. Therefore the purpose of this study is threefold: (1) By focusing on the temporal variation of symmetric local time, we reveal the expression of stochastic differential equations (SDE) for sticky OU processes. Especially, we have studied the sticky OU process as the target model; (2) For the conditional characteristic function of the sticky OU process, We provide important results which helps to get the transfer density; (3) We describe the base interest rate model as a sticky OU process and calculate valuable bond prices under this dynamic.

The remaining parts of the paper are organized as follows. In section 2, sticky OU process and the properties of its solutions are introduced. In section 3, the conditional characteristic function of sticky OU process is shown. In section 4, the bond price when the underlying asset satisfies the sticky OU process is derived. Finally the conclusions are presented in section 5.

2 From OU Process to Sticky OU Process

This section aims to show a display of sticky OU process as a time change version of OU process, and the key is to introduce a time change Brownian motion which serves for the expression of sticky OU process. To see it, let us refer to [40], [32], who characterize the sticky Brownian motion as time change version of the standard Brownian motion. Similar to their idea, let us provide the time change OU process (sticky OU process) as follows. Let B_t be a standard Brownian motion, denote by Y_t a OU process, with initial point y, satisfying

$$Y_t = y + \int_0^t \kappa(\theta - Y_s)ds + \int_0^t \sigma dB_s, \qquad (1)$$

and set a new function

$$r(t) := t + \kappa \hat{L}_t^Y(a),$$

where $\hat{L}_t^Y(a)$ denotes the symmetric local time of Y_t at a. We introduce a time change OU process (i.e., sticky OU process) X_t taking the form of

$$X_t := Y_{r^{-1}(t)},$$

in which $r^{-1}(\cdot)$ is the functional inverse of the strictly increasing function $r(\cdot)$. Note that $r(\cdot)$ is strictly increasing and due to the definition of symmetric local time, it is evidently to derive $\hat{L}_t^X(a) = \hat{L}_{r^{-1}(t)}^Y(a)$, implying

$$r^{-1}(t) = t - \kappa \hat{L}_t^X(a). \qquad (2)$$

On the other hand, since Y_t is an OU process without sticky point, the measure of time set staying at a should be zero, i.e., $\int_0^\eta 1_{\{Y_s=a\}}ds = 0$ for any $\eta > 0$ with respect to Lebesgue measure. We compute

$$\begin{aligned}\int_0^t 1_{\{X_s=a\}}ds &= \int_0^{r^{-1}(t)} 1_{\{Y_s=a\}}dr(s) \\ &= \int_0^{r^{-1}(t)} 1_{\{Y_s=a\}}ds + \kappa \int_0^{r^{-1}(t)} 1_{\{Y_s=a\}}d\hat{L}_s^Y(a) \\ &= 0 + \kappa \int_0^t 1_{\{Y_{r^{-1}(s)}=a\}}d\hat{L}_{r^{-1}(s)}^Y(a) \\ &= \kappa \int_0^t 1_{\{X_s=a\}}d\hat{L}_s^X(a) \\ &= \kappa \hat{L}_t^X(a), \end{aligned} \qquad (3)$$

where the last equality holds because $s \mapsto \hat{L}_s^X(a)$ only increases when $X_s = a$. In the light of (2) and (3), the function $r^{-1}(\cdot)$ as well meets

$$r^{-1}(t) = t - \int_0^t 1_{\{X_s=a\}}ds = \int_0^t 1_{\{X_s\neq a\}}ds.$$

As a result, it follows recalling $X_t = Y_{r^{-1}(t)}$ that

$$\begin{aligned} X_t =& x + \int_0^{r^{-1}(t)} \kappa(\theta - Y_s)ds + \int_0^{r^{-1}(t)} \sigma dB_s \\ =& x + \int_0^t \kappa(\theta - Y_{r^{-1}(s)})dr^{-1}(s) + \int_0^t \sigma dB_{r^{-1}(s)} \\ =& x + \int_0^t \kappa(\theta - X_s)1_{\{X_s\neq a\}}ds + \int_0^t \sigma 1_{\{X_s\neq a\}}dW_s. \end{aligned}$$

The last equality holds by the theorem below.

Theorem 1. *Let B_t be a standard Brownian motion and $r^{-1}(\cdot)$ is defined by (2), then there exists another standard Brownian motion W_t, such that*

$$dB_{r^{-1}(t)} = 1_{\{X_t \neq a\}} dW_t.$$

Proof. Let $\widetilde{W_t}$ be Brownian motion independent of B_t. Define $W_t = B_{r^{-1}(t)} + \int_0^t I_{\{X_s=a\}} d\widetilde{W}_s$. Because B_t and $\widetilde{W_t}$ are independent, the quadratic variation of this object is

$$\langle W \rangle_t = r^{-1}(t) + \int_0^t 1_{\{X_s=a\}} ds = \int_0^t (I_{\{X_s \neq a\}} + 1_{\{X_s=a\}}) ds = t.$$

It is clear that

$$B_{r^{-1}(t)} = \int_0^t 1_{\{X_s \neq a\}} dW_s.$$

Hence we complete the proof. □

Consequently, we can acquire the sticky OU process X_t from standard OU process which satisfies the following SDEs

$$\begin{cases} X_t = x + \int_0^t \kappa(\theta - X_s) 1_{\{X_s \neq a\}} ds + \int_0^t \sigma 1_{\{X_s \neq a\}} dW_s, \\ \int_0^t 1_{\{X_s=a\}} ds = \beta \hat{L}_t^X(a). \end{cases} \tag{4}$$

To see the uniqueness in law for sticky OU (sticky at 0), we cite for two recent work by [41], [42], respectively. More precisely, it can be seen from Theorem 1 in [41], that the sticky OU process has a unique weak solution. However, their sticky point is at 0, while in this paper the sticky point is at $a > 0$. Fortunately, parallel to their idea, we will give the uniqueness theorem for our model.

Theorem 2. *If the sticky OU process satisfies (4), then it has a unique weak solution.*

Proof. To see the uniqueness in law holds for (4), we will undo the time change from the previous part starting with the notation afresh. Suppose X and W solve (4). As part of this hypothesis, we know that X and W are defined on a filtered probability space (Ω, $\mathcal{F}$, $\mathcal{F}_t$, P), both X and W are $\mathcal{F}_t$-adapted, and W is not only a standard Brownian motion with respect to P but also a martingale with respect to $\mathcal{F}_t$. Consider the additive functional

$$T_t = \int_0^t 1_{\{X_s \neq a\}} ds, \tag{5}$$

for $t \geq 0$, and obviously $T_t \uparrow T_\infty$ as $t \uparrow \infty$ where $T_\infty \in (0, \infty]$. Since $t \mapsto T_t$ is increasing and continuous it follows that its (right) inverse $t \mapsto A_t$ defined by

$$A_t = \inf\{s \geq 0 \mid T_s > t\}, \tag{6}$$

is finite for all $t \in [0, T_\infty)$. Note that $t \mapsto A_t$ is increasing and right-continuous on $[0, T_\infty)$. In addition, since $T = (T_t)_{t \geq 0}$ is adapted to $\mathcal{F}_t$ it follows that each A_t is a stopping time with respect to $\mathcal{F}_t$, so that A_t defines a time change with respect to $\mathcal{F}_t$ for $t \in [0, T_\infty)$. Consider the time-changed process

$$Z_t = X_{A_t},$$

for $t \geq 0$. By (4), we have

$$X_t = x + \int_0^t \kappa(\theta - X_s) dT_s + \int_0^t \sigma dM_s, \tag{7}$$

where $M = (M_t)_{t \geq 0}$ is a continuous martingale with respect to $\mathcal{F}_t$ taking the form of

$$M_t = \int_0^t 1_{\{X_s \neq a\}} dW_s,$$

for $t \geq 0$. Note that $t \mapsto \langle M, M \rangle = \int_0^t 1_{\{X_s \neq a\}} ds = T_t$ is constant on each $[A_{s-}, A_s]$ and hence the same is true for $t \mapsto M_t$ whenever $s > 0$ is given and fixed. It follows that M_{A_t} is a continuous martingale with respect to $\mathcal{F}_{A_t}$ and we acquire

$$\langle M_A, M_A \rangle_t = \langle M, M \rangle_{A_t} = T_{A_t} = t,$$

for $t \in [0, T_\infty)$. Using Lévy's characterisation theorem we can therefore conclude that $W_t = M_{A_t}$ is a standard Brownian motion for $t \in [0, T_\infty)$. Moreover, using that $t \mapsto M_t$ is constant on each $[A_{s-}, A_s]$ for $s > 0$, we conclude from (7) that

$$\begin{aligned} Z_t &= x + \int_0^t \kappa(\theta - X_{A_s}) dT_{A_s} + \int_0^t \sigma dM_{A_s} \\ &= x + \int_0^t \kappa(\theta - Z_s) ds + \int_0^t \sigma dW_s, \end{aligned} \tag{8}$$

for $t \in [0, T_\infty)$. Recalling that the stochastic differential equation (1) has a unique strong solution, this shows that Z_t for $t \in [0, T_\infty)$ is a Feller branching diffusion process with drift $\kappa\theta$ having 0 as an instantaneously reflecting boundary point. Moreover, using (4) we see that

$$t = T_t + \int_0^t 1_{\{X_s=a\}} ds = T_t + \beta \hat{L}_t^X(a), \tag{9}$$

from which we find that

$$A_t = T_{A_t} + \beta \hat{L}_{A_t}^X(a) = t + \beta \hat{L}_{A_t}^X(a), \tag{10}$$

for $t \in [0, T_\infty)$. Since $t \mapsto T_t$ is constant on each $[A_{s-}, A_s]$ for $s > 0$ we see from

$$\hat{L}_t^X(a) = \lim_{\varepsilon \downarrow 0} \frac{1}{m((a, a+\varepsilon])} \int_0^t I_{\{a < X_s \leq a+\varepsilon\}} ds,$$

where m is the speed measure of X, that

$$\begin{aligned}\hat{L}^X_{A_t}(a) =& \lim_{\varepsilon\downarrow 0}\frac{1}{m((a,a+\varepsilon])}\int_0^{A_t} 1_{\{a<X_s\le a+\varepsilon\}}ds\\ =& \lim_{\varepsilon\downarrow 0}\frac{1}{m((a,a+\varepsilon])}\int_0^{A_t} 1_{\{a<X_s\le a+\varepsilon\}}dT_s\\ =& \lim_{\varepsilon\downarrow 0}\frac{1}{m((a,a+\varepsilon])}\int_0^{t} 1_{\{a<X_s\le a+\varepsilon\}}dT_{A_s}\\ =& \lim_{\varepsilon\downarrow 0}\frac{1}{m_Z((a,a+\varepsilon])}\int_0^{t} 1_{\{a<X_s\le a+\varepsilon\}}ds\\ =& \hat{L}^Z_t(a),\end{aligned}$$

for $t \in [0, T_\infty)$. Backing into (10), we have

$$A_t = t + \beta\hat{L}^Z_t(a), \tag{11}$$

for $t \in [0, T_\infty)$. Letting $t \uparrow T_\infty$ and using that the diffusion local time process $\hat{L}^Z_t(a)$ of the Feller branching diffusion process Z solving (9) is finite at every finite time, we see that $A_{T_\infty} < \infty$ while by (5) and (6) we see that $A_{T_\infty} = \infty$ whenever $T_\infty < \infty$. This shows that $T_\infty = \infty$ almost surely and consequently the process Z solves (8) for all $t \geq 0$. From (11) we can see that $t \mapsto A_t$ is strictly increasing (and continuous) and hence

$$T_t = A_t^{-1}, \tag{12}$$

is the proper inverse for $t \geq 0$ (implying also that $t \mapsto T_t$ is strictly increasing and continuous). It follows in particular that $A_{T_t} = t$ so that

$$X_t = X_{A_{T_t}} = Z_{T_t}, \tag{13}$$

for $t \geq 0$. Since Z is a unique strong solution to the stochastic differential equation (8), we see from (11)-(13) that X is a well-determined measurable functional of the standard Brownian motion W. This shows that the law of X solving (4) is uniquely determined and the proof of weak uniqueness is complete. □

It is well known that the occupation time set at a of the standard Brownian motion has zero Lebesgue measure. That is, for any $T > 0$

$$\int_0^T 1_{\{W_s=a\}}ds = 0,$$

with probability one. The occupation time set at a of the standard Brownian motion $C^Y_a = \{t \geq 0 : Y_t = a\}$ is topologically a Cantor set with probability one, which implies C^Y_a is a closed nowhere dense set that is its own boundary. However, the sticky Brownian motion will spend positive time at the sticky point. Its occupation time set is still closed and nowhere dense, but it has positive measure with respect to Lebesgue measure. In addition, we mention the infinitesimal generator and its domain of sticky skew process (see, [34]) as follows:

$$\begin{aligned}\mathcal{A}f(x) =& \frac{1}{2}\sigma^2 f^{''}(x) + \kappa(\theta - x)f^{'}(x),\\ Dom(\mathcal{A}) =& \{f \in C(\mathbb{R}) \cap C^2(\mathbb{R}\backslash\{a\}) : \mathcal{A}f \in C(\mathbb{R}),\\ & pf^{'}(a+) - (1-p)f^{'}(a-) = \beta\mathcal{A}f(a)\}.\end{aligned} \tag{14}$$

That is, the domain of definition of $\mathcal{A}$ consists of functions f which are twice continuously differentiable on $\mathbb{R}\backslash\{a\}$. The first derivatives of the functions may be discontinuous at the point in a, but the first derivatives must have left and right limits denoted as $f^{'}(a-)$ and $f^{'}(a+)$ respectively. Although the first and second derivatives lack continuity, $\mathcal{A}f$ is continuous. Lastly, such f satisfies the boundary conditions above and we call equality (14) the sticky boundary condition.

3 Conditional Characteristic Function of Sticky OU Process

This section intends to calculate the conditional characteristic function of sticky OU model X_t, which is defined by

$$\psi(X_{t+\tau}, u, \tau; X_t) = E[\exp(iuX_{t+\tau})|X_t], \qquad \tau = T - t.$$

Theorem 3. *If X_t satisfies the SDE (4), then*

$$\begin{aligned}\psi(X_{t+\tau}, u, \tau, X_t) =& \exp\{C(\tau) + \tilde{D}(\tau)1_{\{X_t\neq a\}}X_t\\ &+ D(\tau)1_{\{X_t=a\}}X_t\},\end{aligned}$$

where

$$\tilde{D(\tau)}1_{\{X_t\neq a\}} = \begin{cases} D^+(\tau), & X_t > a,\\ D^-(\tau), & X_t < a,\end{cases}$$

and with coefficients

$$\begin{aligned}C(\tau) &= iu\theta(1 - e^{-\kappa\tau}) + \frac{i^2u^2\sigma^2}{4k}(1 - e^{-2\kappa\tau}),\\ D(\tau) &= iue^{-\kappa\tau},\\ D^+(\tau) &= \frac{1}{2}D(\tau) + \beta[iu\kappa(\theta - a)e^{-\kappa\tau} + \frac{1}{2}i^2u^2\sigma^2e^{-2\kappa\tau}],\\ D^-(\tau) &= \frac{1}{2}D(\tau) - \beta[iu\kappa(\theta - a)e^{-\kappa\tau} + \frac{1}{2}i^2u^2\sigma^2e^{-2\kappa\tau}].\end{aligned}$$

Proof. Recalling the definition of the conditional characteristic function of X_t at the beginning of this section and applying Tanaka-Meyer formula for sticky model, we have for $X_t \neq a$,

$$d\psi = \Big[-\frac{\partial \psi}{\partial \tau} + \frac{\partial \psi}{\partial X_t}\kappa(\theta - X_t) + \frac{1}{2}\frac{\partial^2 \psi}{\partial X_t^2}\sigma^2 \Big] dt + \frac{\partial \psi}{\partial X_t}\sigma dW_t.$$

Evidently, due to the fact that conditional expectation $\psi(X_{t+\tau}, u, \tau; X_t)$ is a martingale, it means that drift coefficient in $d\psi$ equals to 0, i.e.,

$$-\frac{\partial \psi}{\partial \tau} + \frac{\partial \psi}{\partial X_t}\kappa(\theta - X_t) + \frac{1}{2}\frac{\partial^2 \psi}{\partial X_t^2}\sigma^2 = 0, \quad (15)$$

subject to the boundary condition $\psi(X_T, u, 0|X_T) = \exp(iuX_T)$ when $\tau = 0$ and the sticky boundary condition. To solve the last equation, we apply the method of undeterminated coefficients. Suppose that the display of the solution to (15) takes the form of

$$\psi(X_{t+\tau}, u, \tau, X_t) = \exp\{C(\tau) + \tilde{D}(\tau)1_{\{X_t \neq a\}}X_t + D(\tau)1_{\{X_t = a\}}X_t\}, \quad (16)$$

with the expression and the boundary condition $\tilde{D(\tau)}1_{\{X_t \neq a\}} = D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}}$, $C(0) = 0$ and $D^+(0)1_{\{X_t > a\}} + D^-(0)1_{\{X_t < a\}} = iu$.

Based on (16), the first and second order partial derivatives are shown as below

$$\frac{\partial \psi}{\partial \tau} = \psi[C^{'}(\tau) + (D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}})^{'}X_t],$$
$$\frac{\partial \psi}{\partial X_t} = \psi(D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}}),$$
$$\frac{\partial^2 \psi}{\partial {X_t}^2} = \psi(D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}})^2.$$

Substituting these equations in (15) implies

$$\begin{aligned}&-C^{'}(\tau) + \kappa\theta[D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}}] + \frac{1}{2}\\ &\sigma^2[D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}}]^2 - [(D^+(\tau)1_{\{X_t > a\}}\\ &+ D^-(\tau)1_{\{X_t < a\}})^{'} + \kappa(D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}})]X_t\\ &= 0.\end{aligned}$$

Note that the above equation holds for any X_t, thus we have

$$\begin{cases} (D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}})^{'} + \kappa(D^+(\tau)1_{\{X_t > a\}} \\ +D^-(\tau)1_{\{X_t < a\}}) = 0, \\ -C^{'}(\tau) + \kappa\theta[D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}}] \\ +\frac{1}{2}\sigma^2[D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}}]^2 = 0. \end{cases} \quad (17)$$

For the first equality in (17), we get

$$\frac{d(D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}})}{D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}}} = -\kappa d\tau.$$

thus

$$\ln[D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}}] = -\kappa\tau + c.$$

Recalling the boundary condition $D^+(0)1_{\{X_t > a\}} + D^-(0)1_{\{X_t < a\}} = iu$,

$$D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}} = iue^{-\kappa\tau} := D(\tau), \quad (18)$$

Obviously, by the boundary condition and the expression of $D(\tau)$, we compute the result for $C(\tau)$.For convenience, we write the infinitesimal generator as

$$\mathcal{A}\psi(x) := \frac{\partial \psi}{\partial x}\kappa(\theta - x) + \frac{1}{2}\frac{\partial^2 \psi}{\partial x^2}\sigma^2.$$

Moreover, the infinitesimal generator and its domain of sticky OU process provide that

$$\frac{1}{2}\frac{\partial \psi}{\partial X_t}\Big|_{X_t = a+} - \frac{1}{2}\frac{\partial \psi}{\partial X_t}\Big|_{X_t = a-} = \beta\mathcal{A}\psi\big|_{X_t = a} = \beta\frac{\partial \psi}{\partial \tau}\Big|_{X_t = a},$$

where the last equation comes from (15). Thanks to the sticky boundary condition (14), we derive another important condition for $D^+(\tau)$ and $D^-(\tau)$ by

$$D^+(\tau) - D^-(\tau) = 2\beta[iu\kappa(\theta - a)e^{-\kappa\tau} + \frac{1}{2}i^2u^2\sigma^2e^{-2\kappa\tau}]. \quad (19)$$

Combining (19) with (18) gives the results of $D^+(\tau)$ and $D^-(\tau)$. At last, after supplementing the result for $X_t = a$ we finish this proof. □

When $\beta = 0$, the sticky process reduces to the standard process without sticky point and the result will not be piecewise. In addition, from (18) and (19), we easily derive $D^+(\tau)1_{\{X_t > a\}} = D^-(\tau)1_{\{X_t < a\}}$ and $D^+(\tau)1_{\{X_t > a\}} + D^-(\tau)1_{\{X_t < a\}} = D(\tau)$. Sometimes it is difficult to obtain the density function directly. Alternatively, it seems more tractable to gain the characteristic function and one can compute the transition density and the conditional moment with the relationship given by

$$E[X_t^n] = (-i)^n\psi^{(n)}(X_t, 0),$$

and

$$f(X_t) = \frac{1}{2\pi}\int_{-\infty}^{+\infty} \exp(-iuX_t)\psi(X_t, u)du,$$

respectively. So far, we have acquired the explicit expressions of the conditional characteristic function of the sticky OU process X_t. Actually, it is not effortless to get the transition density for this general process, which, as a result, prompts us to focus on the characteristic function. Next section, we devote ourself to the application in bond pricing under sticky OU process.

4 Bond Pricing under Sticky OU Process

In this section, we are interested in deriving the bond price when the underlying asset r_t, and suppose r_t is the only variable that affects the price of the bond. Suppose that the finance market considered in our paper is arbitrary-free and the default-free bond price at time t is denoted by $P(t,T)$ or $P(t,\tau)$, where T is the maturity and $\tau = T - t$ is the bond's term. Let $P(r_t,\tau)$ be the bond price based on r_t with the maturity T at time t. By Tanaka-Meyer formula, we have,

$$\begin{aligned} dP &= [\frac{\partial P}{\partial t} + \kappa(\theta - r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2}]dt + \frac{\partial P}{\partial r}\sigma dW_t \\ &= \mu_p(r,\tau)Pdt - \sigma_p(r,\tau)PdW_t, \end{aligned}$$

where

$$\mu_p(r,\tau) := [\frac{\partial P}{\partial t} + \kappa(\theta - r)\frac{\partial P}{\partial r} + \frac{1}{2}\sigma^2\frac{\partial^2 P}{\partial r^2}]/P, \tag{20}$$

$$\sigma_P(r,\tau) := -\frac{\partial P}{\partial r}\sigma/P. \tag{21}$$

What follows next is our main result about the bond price under sticky OU dynamic.

Theorem 4. *Suppose that the underlying zero coupon interest rate satisfies* (4) *and λ is the sharpe index in the modern market which is arbitrary-free. Then the bond price $P(t,\tau)$ of the zero coupon interest rate with the maturity time T is represented by*

$$P(r_t,\tau) = \exp\{A(\tau) + \tilde{B(\tau)}1_{\{r_t\neq a\}}r_t + B(\tau)1_{\{r_t=a\}}r_t\},$$

where

$$\tilde{B(\tau)}1_{\{r_t\neq a\}} = \begin{cases} B^+(\tau), & r_t > a, \\ B^-(\tau), & r_t < a, \end{cases}$$

and with the coefficients

$$\begin{aligned} A(\tau) =& [-B(\tau) - \tau](\theta + \lambda\frac{\theta}{\kappa} - \frac{1}{2}\frac{\sigma^2}{\kappa^2}) - \frac{\sigma^2 B^2(\tau)}{4\kappa}, \\ B(\tau) =& \frac{e^{-\kappa\tau} - 1}{\kappa}, \\ B^+(\tau) =& \frac{1}{2}B(\tau) + \beta[\frac{1}{2}\sigma^2 B^2(\tau) + \kappa(\theta - a)B(\tau)], \\ B^-(\tau) =& \frac{1}{2}B(\tau) - \beta[\frac{1}{2}\sigma^2 B^2(\tau) + \kappa(\theta - a)B(\tau)]. \end{aligned}$$

Proof. In the bond pricing theory, if a bond market is arbitrary-free, the sharpe ratio of trading bonds with different terms should be equal.Vasicek (1977) assumed that the market price of the risk is equal to a constant λ i.e., for any bond term τ

$$\frac{\mu_p(r,\tau) - r}{\sigma_p(\tau)} = \lambda, \tag{22}$$

in our paper, we introduce the same ratio in (22) .With the expressions of (20), (21) and (22),and substitute $\frac{\partial P}{\partial t} = -\frac{\partial P}{\partial \tau}$,we establish

$$\frac{1}{2}\frac{\partial^2 P}{\partial r^2}\sigma^2 + [\kappa(\theta - r) + \lambda\sigma]\frac{\partial P}{\partial r} - \frac{\partial P}{\partial \tau} - rP = 0. \tag{23}$$

with the boundary condition $P(r_t, 0) = 1$.
Similar idea to section 3, we naturally suppose that the solution to (23) takes the form of

$$P(r_t,\tau) = \exp\{A(\tau) + \tilde{B(\tau)}1_{\{r_t\neq a\}}r_t + B(\tau)1_{\{r_t=a\}}r_t\},$$

with the boundary condition
$\tilde{B(\tau)}1_{\{r_t\neq a\}} = B^+(\tau)1_{\{r_t>a\}} + B^-(\tau)1_{\{r_t<a\}}$
and $A(0) = B^+(0)1_{\{r_t>a\}} + B^-(0)1_{\{r_t<a\}} = 0$.
results in,

$$\begin{aligned} \frac{\partial P}{\partial \tau} &= P[A'(\tau) + (B^+(\tau)1_{\{r_t>a\}} + B^-(\tau)1_{\{r_t<a\}})'r_t], \\ \frac{\partial P}{\partial r} &= P(B^+(\tau)1_{\{r_t>a\}} + B^-(\tau)1_{\{r_t<a\}}), \\ \frac{\partial^2 P}{\partial r^2} &= P(B^+(\tau)1_{\{r_t>a\}} + B^-(\tau)1_{\{r_t<a\}})^2. \end{aligned}$$

Substitute these equations into (23), thus we get

$$\begin{aligned} &- A'(\tau) + \frac{1}{2}\sigma^2(B^+(\tau)1_{\{r_t>a\}} + B^-(\tau)1_{\{r_t<a\}})^2 \\ &+ (\kappa\theta + \lambda\sigma)(B^+(\tau)1_{\{r_t>a\}} + B^-(\tau)1_{\{r_t<a\}}) \\ &- [\kappa(B^+(\tau)1_{\{r_t>a\}} + B^-(\tau)1_{\{r_t<a\}}) \\ &+ (B^+(\tau)1_{\{r_t>a\}} + B^-(\tau)1_{\{r_t<a\}})' + 1]r_t \\ &= 0, \end{aligned}$$

which holds for arbitrary r_t. It suggests that

$$\begin{cases} \kappa(B^+(\tau)1_{\{r_t>a\}} + B^-(\tau)1_{\{r_t<a\}}) \\ +(B^+(\tau)1_{\{r_t>a\}} + B^-(\tau)1_{\{r_t<a\}})' + 1 = 0, \\ -A'(\tau) + \frac{1}{2}\sigma^2(B^+(\tau)1_{\{r_t>a\}} + B^-(\tau)1_{\{r_t<a\}})^2 \\ +(\kappa\theta + \lambda\sigma)(B^+(\tau)1_{\{r_t>a\}} + B^-(\tau)1_{\{r_t<a\}}) = 0. \end{cases}$$

Parallel to the calculation in section 3 and recalling the infinitesimal generator and its domain for f, it follows that

$$\begin{aligned} &\frac{1}{2}\frac{\partial P}{\partial r_t}\Big|_{r_t=a+} - \frac{1}{2}\frac{\partial P}{\partial r_t}\Big|_{r_t=a-} = \beta\mathcal{A}P\big|_{\{r_t=a\}} \\ &= \beta\Big[-\lambda\sigma\frac{\partial P}{\partial r}\Big|_{\{r_t=a\}} + \frac{\partial P}{\partial \tau}\Big|_{\{r_t=a\}} + aP\big|_{\{r_t=a\}}\Big], \end{aligned}$$

where the last equation comes from (23). By calculation,we get the follows equations

$$A(\tau) = [-B(\tau) - \tau](\theta + \lambda\frac{\theta}{\kappa} - \frac{1}{2}\frac{\sigma^2}{\kappa^2}) - \frac{\sigma^2 B^2(\tau)}{4\kappa},$$

$$B^+(\tau)1_{\{r_t>a\}} + B^-(\tau)1_{\{r_t<a\}} = \frac{e^{-\kappa\tau} - 1}{\kappa} =: B(\tau),$$

$$B^+(\tau)1_{\{r_t>a\}} - B^-(\tau)1_{\{r_t<a\}}$$

$$= 2\beta[\frac{1}{2}\sigma^2 B^2(\tau) + \kappa(\theta - a)B(\tau)].$$

Again, if we let $\beta = 0$, the sticky boundary disappears and the sticky OU process reduces to the OU process. After supplementing the result for $r_t = a$, we complete this proof. □

5 Numerical Results

In this section, we want to provide the numerical results for the displays of bond price under sticky OU process with different sticky coefficients β. More precisely, we set $\kappa = 1$, $\theta = 0.04$, $\sigma = 0.2$, $\lambda = 0.5$, $r = 0.03$, $a = -0.04$ as the common parameters. Noted that $a = -0.04$ is the sticky point which later causes some interesting analysis for sticky phenomenon. In addition, in the following three figures, the bond's τ is considered with respect to three different conditions, respectively. In each of three figures, different sticky coefficients are further discussed.

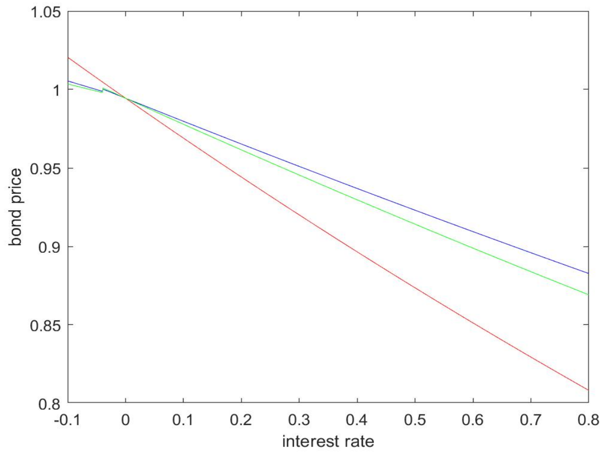

Fig. 4: Bond price under sticky OU process with different sticky coefficients β when $\tau = 0.3$. The red, blue, green lines represent $\beta = 0$, $\beta = 1$, $\beta = 2$, respectively.

Figure 4 shows the displays of the bond price in the case of $\tau = 0.3$. Usually, bond price decreases as underlying interest rate increases. The red line

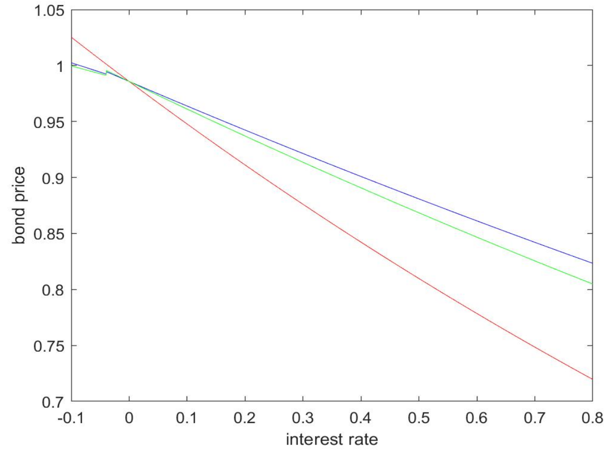

Fig. 5: Bond price under sticky OU process with different sticky coefficients β when $\tau = 0.5$. The red, blue, green lines represent $\beta = 0$, $\beta = 1$, $\beta = 2$, respectively.

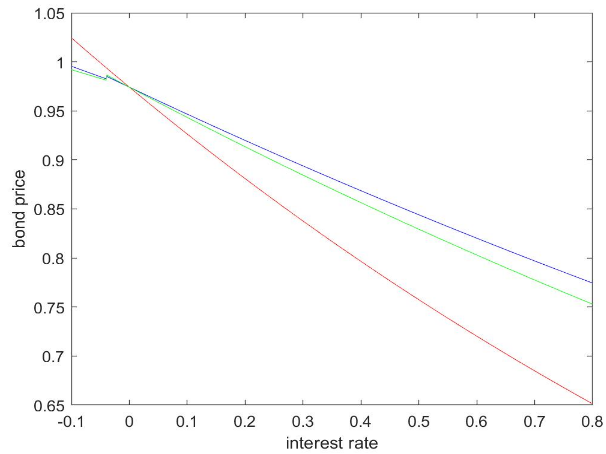

Fig. 6: Bond price under sticky OU process with different sticky coefficients β when $\tau = 0.7$. The red, blue, green lines represent $\beta = 0$, $\beta = 1$, $\beta = 2$, respectively.

represents the classical bond price with respect to sticky coefficient $\beta = 0$. It is interesting to see that around sticky point $a = 0.04$, the bond prices exhibit different behaviors. With a bigger sticky coefficient β, the underlying interest rate will spend more time at a, leading to an aggregation phenomenon and such phenomenon obviously influences the bond price which increases weakly with respect to the ``aggregated" interest rate. However, once the underlying interest rate goes through the sticky point, the bond price will decrease strongly compared with the smaller sticky coefficients.

As the study, [31], said that for a corporation having a takeover offer at 10. The stock price is then likely to spend a great deal of time precisely at 10 but is not constrained to stay at 10. Thus 10 would be a sticky point for the solution of the stochastic differential equation that describes the stock price. For interest rate, we also take sticky phenomenon into consideration. Suppose an interest rate is modelled by the sticky OU process. Then it is possible for bond price to have more choice at the sticky point because the underlying interest rate will spend more time at such fascinating point. But once interest rate passing sticky point, the bond price will exhibit normal principle immediately regardless of the sticky phenomenon. Similar analysis applies to Figure 5 and Figure 6 for $\tau = 0.5$ and $\tau = 0,7$, respectively.

For different bond's term τ, it is evidently to observe the blue lines from Figure 4, Figure 5 and Figure 6, to see that the bond price decreases as the bond's term increases. This coincides with the classical results in bond pricing theory. Also note that the bond price can be greater than one because of the negative interest rate under sticky OU model.

6 Conclusion

In this study, we delve into the theoretical underpinnings and financial applications of sticky Ornstein-Uhlenbeck (OU) processes. To establish the existence and uniqueness of solutions for the sticky OU process, we employ an innovative time transformation technique that transforms standard Brownian motion into sticky Brownian motion. This transformation leverages the properties of symmetric local time to illuminate the asymptotic behavior of sticky boundaries, thereby facilitating the construction of the sticky OU process from its standard counterpart. In our analysis of conditional characteristic functions and bond prices, we harness the power of the martingale property and Sharpe ratio. By assuming an exponential form for the solution, we meticulously solve the governing equations, leading to insightful results. Notably, our work significantly contributes to expanding pricing applications leveraging sticky behavior under generalized conditions, suggesting that sticky processes may offer a more nuanced representation of real-world market dynamics. Looking ahead, we plan to conduct in-depth research on the relationship between bond prices and sticky components under other conditions. In addition, we also plan to study pricing issues for other derivatives, such as options. These studies will provide us with a deeper market understanding and pricing strategies.

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Declaration of Generative AI and AI-assisted technologies in the writing process

During the preparation of this work the authors used [yiyan.baidu] in order to [improve the readability and language of our manuscript]. After using this tool/service, the authors reviewed and edited the content as needed and takes full responsibility for the content of the publication.

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Contribution of Individual Authors to the Creation of a Scientific Article (Ghostwriting Policy)

Haoyan Zhang proposed the idea of the method, provided the numerical results and checked the correctness of the manuscript.

Yece Zhou proved the existence and uniqueness to the sticky OU process as well as the conditional characteristic function.

Yingxu Tian computed the bond price, found the financial data and wrote the article and polish the language.

Sources of Funding for Research Presented in a Scientific Article or Scientific Article Itself

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Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this paper.