Framework for Small Traveling Salesman Problems

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Abstract: We study small traveling salesman problems (TSPs) because current quantum computers can find optional solutions for TSPs with up to 14 cities. Also, we study small TSPs because TSPs have been recommended to be benchmarks to measure quantum optimization on all types of quantum hardware. This means comparisons of quantum data about small TSPs. We extent previous numerical results that were reported in “Small Traveling Salesman Problems” for 6, 8 and 10 cities. The new results in this paper are for 10 – 14 cities in symmetric TSPs. The data for this new range of cities is consistent with the previous data and can be the basis for estimates of results from quantum computers that are upgraded to handle more than 14 cities. The work and analysis suggest two conjectures that we discuss. The paper also contains an annotated survey of recent publications about TSPs.

Keywords: Traveling salesman problem, optimal tour, combinatorial analysis, discrete optimization, quantum annealing, quantum computer

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1 Introduction

Research interest in the traveling salesman problem (TSP) has been stimulated by demonstrated ability to find optimal solutions on quantum processors [28] and by questions such as: What are the characteristics that distinguish easy to solve TSPs from those that are difficult to find an optimal solution on a quantum computer? Additional interest in TSPs has come from proposals to use TSPs as benchmarks for quantum optimization on various hardware [9, 32, 33].

Due to its many applications and computational complexity, the worldwide opinion of the TSP has escalated from an obscure novelty in the 1960’s to a leading example of combinatorial optimization problems. Reference [1], which has prominent contributors and editors, was instrumental in this transition. Textbooks [2 - 5] contributed to the rise. of the TSP.

The current paper is a continuation of [6] where the structure of TSPs for 6, 8 and 10 cities is examined. We observe similar structure in the current study for 10 – 14 cities and show the supporting data in Tables 3 and 4. Interestingly, Table 3 shows without exception that as the number of cities increases, the number of optimal tours increases.

The effort for [6] was motivated by the need to have reference TSPs for comparison to the results of the D-Wave quantum solver that has about 2,000 qubits and can optimally solve TSPs with up to about 8 cities. An upgraded D-Wave quantum solver with about 5,600 qubits can optimally solve TSPs with up to 14 cities. This improvement stimulated the current study to understand the nature of optimal solutions. To obtain an optimal solution on D-Wave’s upgraded quantum processor, the size of the TSP is limited to about 14 cities due to difficulties embedding all-to-all connections of cities. Therefore, in this study we report about optimal solutions of TSPs with at most 14 cities. The technique in [32] to solve the TSP is limited to about 8 cities.

We define the TSP. Given a set of cities and the distance between each pair of cities, the TSP asks for a shortest route that visits each city once and returns to the starting city. A shortest route is called an optimal tour. If for each pair of cities (A, B), the distance from city A to city B is the same as the distance from city B to city A, then the TSP is called symmetric. The term TSP includes those that are symmetric and those that are not symmetric. Initial work defining the TSP on a quantum annealing computer was published in [7, 8, 30].

We studied symmetric and non-symmetric TSPs in [6] without distinction. References [9] and [10] recommend symmetric TSPs as a benchmark for quantum optimization problems and disqualify non-symmetric TSPs as benchmarks. Therefore, the current study is only about symmetric TSPs. This is very significant because there are extremely few random TSPs that are symmetric compared to the number that are non-symmetric.

Since we are interested primarily in an optimal answer, our work does not consider hybrid solvers (quantum and digital combined, each doing part of the solution) because their analog nature usually produces answers that are not optimal. We are interested in the total number of optimal tours and the frequency of occurrence because quantum results can contain several minimum energy solutions that may be near-optimal or optimal tours. This gives the salesman more than one option for a tour.

We outline the contents of this paper. The next Section contains insights and two conjectures. Section 3 contains the settings for the parameters. Section 4 has the data generated in the study. Future studies of the TSP are recommended in Section 5. Section 6 is an annotated survey of recent, relevant articles, mostly about the TSP. Conclusion and results are in Section 7.

2 Open Questions and Conjectures

Symmetric TSPs are beginning to be used to benchmark quantum algorithms and quantum hardware for quantitative optimization problems [27]. This effort is generating new questions about TSPs. Some of them are: (i) How many shortest routes does a TSP have? (ii) What is the gap between the length of a shortest route and the length of a next-to-shortest route for TSPs?

A conjecture related to Question (i): If a TSP has few shortest routes, then adverse quantum effects are likely to occur that curtail solving the TSP on a quantum machine. A conjecture related to Question (ii): If the gap for a TSP is small, then adverse quantum effects are likely to occur. The adverse quantum effects for the TSP are excessive time to solve, failure to return a route, and inability to find a shortest route. We expect that answers to the questions for TSPs can help predict outcomes for quantitative optimization problems.

A study has begun to quantify TSP characteristics by examining TSPs with 6 cities [11]. The small number of qubits at that time limited the size of the TSPs that could be examined. Now an increase in the number of qubits requires larger TSPs. In the current paper we show results for TSPs with 10 – 14 cities with data collected from classical processors.

It is widely recognized that quantum computers are analog devices that may deviate from the theory of an algorithm [12]. This difficulty, coupled with numeric imprecision, caused some quantum calculations to miss optimal solutions for TSPs. For some TSPs the D-Wave quantum computer could not distinguish between an optimal solution and a close-to-optimal solution.

3 Methodology and Parameter Settings

Table 1 lists the parameters and their settings in [6] and the current paper.

Table 1. Settings for the parameters in two studies

Parameter	Reference [6] Setting	Current Paper Setting
Number of Cities	6, 8, 10	10, 11, 12, 13, 14
Number of TSPs studied	5,000 for each number of cities	200 for 10 cities 100 for 11 and 12 cities 51 for 13 cities 5 for 14 cities
Distances between cities	Random integers ∈ {1, 2, …, 21}	Random integers ∈ {1, 2, …, 21}
Type of TSP	Did not distinguish between symmetric and non-symmetric	Symmetric
Solution algorithm	Python exact, examine all tours	Python exact, examine all tours

Next, we show two examples of TSPs.

Example 1 is a symmetric TSP on 6 cities. Let the cities be designated A, B, C, D, E, F. Let a distance matrix X be given that contains the distance between each pair of cities. Since the TSP is symmetric, X is a symmetric matrix, i.e., upper triangular. The diagonal elements of X have no role in the TSP.

Example 2 is a symmetric TSP on 8 cities. We can describe Example 2 as an expansion of Example 1. Two additional designations are needed for cities and additional distances are needed to expand X.

We comment about the distances restricted to 1, 2, …, 21 in Table 1. A D-Wave implementation transcribes coefficients to the interval [-10, 10]. When the original coefficients are integers 1, 2, …, 21, then the D-Wave mapping to the [-10, 10] has the greatest accuracy because it is a 1 to 1 mapping of integers onto integers [13]. TSPs with all distances 1 and 2 are NP-complete and have been studied in [14].

Since it takes significantly longer to test TSPs with a larger number of cities, we scaled down the total number tested as the number of cities increased.

4 Data Analysis and Findings

Table 2 shows the distribution of random, symmetric TSPs according to the number of optimal tours. Since the TSPs are symmetric, a tour and its inverse have the same length, which means the number of optimal tours is an even integer. The data for 14 cities is weak since the sample size is very small.

Table 2. Distribution of random, symmetric TSPs for 10 to 14 cities

	10 Cities	11 Cities	12 Cities	13 Cities	14 Cities
2 Optimal Tours	159	75	67	32	3
4 Optimal Tours	32	16	21	12	
6 Optimal Tours	8	9	8	5	1
8 Optimal Tours	1		1		1
10 Optimal Tours			3		
12-16 Optimal Tours					
18 Optimal Tours				1	
20 Optimal Tours				1	
Total Number of TSPs	200	100	100	51	5
Number of Optimal Tours	502	268	304	180	20

Table 3 is Table 2 normalized to 100 TSPs for each number of cities.

Table 3. Normalized distribution of random, symmetric TSPs for 10 to 14 cities

	10 Cities	11 Cities	12 Cities	13 Cities	14 Cities
2 Optimal Tours	79.5	75	67	64	60
4 Optimal Tours	16	16	21	24	
6 Optimal Tours	4	9	8	10	20
8 Optimal Tours	0.5		1		20
10 Optimal Tours			3		
12-16 Optimal Tours					
18 Optimal Tours				2	
20 Optimal Tours				2	
Number of Optimal Tours	251	268	304	360	400

Table 4 shows data for the distribution of the average length of optimal tours. We did not compute this for TSPs with 14 cities and for TSPs with 13 cities that have 18 and 20 optimal tours due to a small number of TSPs.

Table 4. Distribution of average length of optimal tours for 10 to 13 cities

	10 Cities	11 Cities	12 Cities	13 Cities
2 Optimal Tours	44.2	44.8	48.4	45.8
4 Optimal Tours	48.2	47.3	51.4	46.8
6 Optimal Tours	47.0	49.9	50.4	44.0
8 Optimal Tours	53.0		52.0	
10 Optimal Tours			60.0	
Number of TSPs	200	100	100	49

Results that are like those in Table 4 are shown for TSPs with 6, 8 and 10 cities in Figures 1 – 3 of [6].

Tests have been run on the D-Wave quantum machine that solve small, symmetric TSPs exactly. In addition to our work, reference [28] reports several TSPs solved exactly.

5 Research Directions in the Future

When D-Wave upgrades its array of qubits in its quantum processor, then the size of TSPs that can be processed without heuristics or hybrid methods is expected to increase. This number of cities will need to be determined. Then a study like the current one should be undertaken for the new numbers of cities beyond 14 and for 13 & 14 cities for overlap. The classical software Concorde [15] is recommended to establish a baseline, since it is the gold standard for solving symmetric TSPs.

The average gap between the length of an optimal tour and the length of the next shortest tour can be determined for various categories of TSPs. Is there a correlation between the size of the gap and easy or difficult to solve with a quantum algorithm, i.e., do large gaps correspond to 'easy to solve' on a quantum processor? 'Easy to solve' can be described numerically by the time to solve, accuracy of the solution, and/or a percentage of the optimal tours found. This leads to generating and assembling TSP data about the two conjectures in the Preface. Based on future data, we can begin to decide the validity of the conjectures and how to quantify them.

A major challenge is the need to deal with more sophisticated, real-world problems.

Let the number of cities be fixed. What sample size is needed to have X% confidence that all symmetric TSPs have a property of the samples? Are the sample sizes in this paper adequate?

Lastly, we comment that large companies including Google, Microsoft, IBM, D-Wave and [34] are working to improve their quantum hardware and algorithms in order to secure their marketplace for this new technology in the business world.

6 Recent Publications Related to Experimental Results about TSPs

In this section we provide fresh insights, breakthroughs and collaborations from the literature.

Reference [16] categorizes an extensive list of references in their Tables 9 - 11. The TSP is included in the analysis and comparisons.

The authors of [17] introduce an improved version of quantum annealing to handle local optimal results when solving the TSP on a D-Wave processor. Since there are difficulties in the theory, the experimental results are remarkable.

The quantum modeling techniques in [18] are designed for variants of the TSP. It may be interesting to recast them for basic TSPs and test them on a D-Wave quantum processor for 6-city problems. This is the smallest number of cities that can have two distinct loops; in this case each loop has three cities.

Paper [19] uses a logical embedding of a TSP in the qubit array of D-Wave's 2,000 and 5,000 quantum processors. The conclusions agree with what is known previously. In general, a TSP with at most 8 cities can be embedded in D-Wave's 2,000 qubit machine

[20 Section 5.1] and a TSP with at most 13 cities can usually be embedded in D-Wave's 5,000 qubit machine [10 Section 2].

The remarkable TSP results that are claimed in [20] need an independent investigation that repeats the experiments.

The authors of paper [21] investigate the performance of two classical and two quantum optimization algorithms to solve the TSP. The overall conclusion of the authors is that current classical devices significantly outperform the IBM quantum devices.

Publication [22] has the same authors and topic as [21]. In [22] results on the TSP for two classical optimization algorithms are compared with one quantum algorithm on a range of IBM quantum devices. There is insufficient information about the attributes of the TSPs in the experiments. The overall conclusion of the authors is that the classical optimization techniques outperform the quantum methods in both computational time and solution quality.

We call attention to paper [23] because it has similarities to the current paper. In [23] pairs of integers from a 100 x 100 square are candidates for cities. Randomly N pairs are chosen from the square for the cities of a TSP. The distance between cities is the Euclidean distance rounded down to an integer. According to [23 page 3], the plan is to address four questions with empirical data. 1. What is the distribution of the distances? 2. What is the distribution of the lengths of tours for instances of the same size? 3. What is the distribution of lengths of optimal tours for instances of the same size? 4. How difficult is it to solve an instance? Results for 1 – 3 are in the paper but apparently not for 4.

Articles [24] and [25] claim to have a technique to reduce the number of variables in a D-Wave QUBO for the TSP. The result is expected to be larger TSPs processed by a quantum annealer without heuristics, resulting in better quality solutions at the expense of longer quantum execution time [25 Table 1]. The papers lack comparisons of experimental data for solving TSPs with and without this enhancement. We recommend that the data be collected for random, symmetric TSPs with various numbers of cities on both the D-Wave 2000 and 5000 processors. It is recommended that the data be presented in a table that identifies the quantum processor, the TSP, the number of cities, the length of an optimal tour, length of the shortest tour (found by the quantum processor) with and without the enhancement and number of qubits used with and without the enhancement. Also, time for the quantum processing unit with and without the enhancement is a useful comparison.

The authors of [26] have developed a computational method for generating metric TSPs that are hard for Concordia [15] to solve, i.e., a long runtime is needed by Concorde to find an optimal tour.

Using simulated Ising quantum software with all-to-all connections, the eleven authors of [31] show details to find an optimal solution for a 9-city, symmetric TSP. Based on that technique they experimentally solve a 70-city TSP. The simulation operates with high energy efficiency compared to several quantum computers, including D-Wave's 2,000 qubit Ising machine.

The researchers of paper [33] use the TSP on gate-based NISQ hardware to show that their quantum solution algorithm is superior to digitized quantum annealing and the quantum approximate optimization algorithm. It appears that [33] reports success for a 3-city TSP solved on an IBM superconducting machine and for a 3-city TSP and a 4-city TSP solved on an IonQ trapped-ion machine. These very small TSPs are trivial to solve. Results for 100 random 6-city TSPs would be much more useful.

7 Conclusions

The current study examines symmetric TSPs that have 10 – 14 cities with distances between cities restricted to random integers in the interval 1 to 21. The conclusions in a previous study [6] are essentially the same in the current study. We describe them. Let n be an integer and $6 \leq n \leq 13$. Then according to [6] and Tables 3 and 4 for large collections of n-city TSPs, most likely the number of optimal tours per TSP and the average length of an optimal tour increase together. This data is the first connection between the number of optimal tours per TPS and the average length of an optimal tour.

The data in the last line of Table 3 shows for 10 – 14 cities that the number of optimal tours increases as the number of cities increases. This result has no exceptions. Recalling that each column of Table 3 is normalized to 100 TSPs. The rate of increase of optimal tours across Table 3 is very slow compared to the factorial rate of increase of tours.

Looking down the columns of Table 3, the conclusion is that as the number of optimal tours increases the frequency of optimal row tours decreases, except for two anomalies, one for 12 cities and the other for 14 cities.

Table 4 indicates that for 2 – 6 optimal tours as the number of cities increases from 10 to 11 and from 11 to 12, the average length of an optimal tour increases. The only exception occurs for 4 optimal tours transitioning from 10 cities to 11 cities.

Looking down the columns of Table 4 we observe that in most cases (8 of 11) as the number of optimal tours increases, the length of an optimal tour increases. The three exceptions occur between 4 and 6 optimal tours.

Similar results shown are anticipated for quantum annealing solutions of large numbers of TSPs.

In conclusion, we point to an outstanding lecture about the TSP [29].

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