On Models of Population Evolution of Three Interacting Species

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Abstract: In this paper, we first analyzed several basic population dynamics models interpreting the relationships between three species. These are the May-Leonard model with three competitors, some prey-predator models of three-species and a prey-predator model with a super-predator. Subsequently, in our work, we proposed a new three-species model consisting of a prey, a predator and a super-predator, including some important assumptions such as competition, self-defense and infected prey. We examined the various equilibrium points of proposed model, and determined the conditions for extinction and survival of species in the long term. Finally, we performed numerical illustrations using Maltlab software to corroborate the theoretical results.

Key-Words: Prey-predator model, population dynamics, stability, persistence, extinction, self defense, infectious.

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1 Introduction

Prey-predator interaction is one of the most widely discussed topics in the field of population dynamics, [1, 2, 3]. Early research focused on the predation relationship between a predator and its prey. In these studies, The authors focused on the fluctuation of population densities during interactions. Several other types of relationships (competition, cooperation, parasitism...) between species have also been studied. The authors in [4], for example, analyze different types of prey-predator relationships (competition, cooperation, competition and cooperation), providing numerical illustrations to corroborate theoretical calculations. Subsequently, the relationships between predators, their preys and the biotope were examined in several studies. Some authors reveal the effects that climate change can have on interactions between preys and predators in various ecosystems; others show that habitat may cause disruption of interactions between preys and predators, [5]. It's a subject of crucial importance, given our interest in the fate of the creatures that surround us and provide us with resources, [6, 7]. To help preserve certain essential resources, some studies have focused on the problems of harvesting and hunting, with a view to finding optimal conditions for the use of harvested resources. The results of these studies define the biomass stocks to be preserved in order to guarantee the regeneration of these resources, [3, 8]. A number of studies have been carried out on the relationship between predators and their preys, including infected individuals. During prey-predator interactions, a phenomenon very often occurs that is detrimental to the predator population. In their desire to escape from predators, prey use several means to defend themselves (this is the case of buffaloes and giraffes that often bear fatal blows to their predators). Refuge is one of these means, as the authors in [4], discuss. Some prey defend themselves vigorously against predators, often causing enormous damage to the predator population. Some studies also take into account group hunting and defense. In [9], for example, the authors examine the evolution of populations of prey that defend themselves in groups and generalist predators that hunt co-

operatively; (this is the case, for example, of hyenas, which generally hunt in groups). On the other hand, the consumption of certain infected prey can lead to disease in the predator population, resulting in high reduction rates. In fact, consumption of infectious prey is detrimental to predators. In [10], the authors discuss the health benefits of a healthy diet. In this paper, we analyze several models of interaction between three species. Thus we propose and study a new model of interaction between three species. Our model incorporates generalist predators, wich have multiple alternative food sources and can enhance their fitness by using theses resources. We also consider that some prey species possess the ability to defend themselves collectively, and these defensive actions negatively impact the predator population. specifically, such actions can lead to losses within the predator population, especially affecting younger predators which are often injured. we account for the fact some prey are infected, leading to their destruction and that the predators consuming these infected prey (whether dead or alive) also suffer from the infection. We represent by a rate τ all the harmful actions (infection, defenses) of the prey on the predators.
Our work is organized as follows. The section 2 which examines a number of models of population evolution of three species in a given environment, contains several subsections. We start by reviewing key essential concepts for our analysis. Next, we present and analyze the May-Leonard model with three competitors and then, we discuss several prey-predator models with three species. These models include prey-predator model with two preys species and one predator, the prey-predator-super predator model and the prey-predator with prey harvesting. For each model, we provide examples of natural interactions, study equilibrium points and perform numerical simulations.
Finally, in Section 3, we propose a new three-species prey-predator model that takes into account competition, self-defence and the presence of infected prey. We examine different equilibrium points of proposed model, determine the conditions for extinction and long-term survival and conduct numerical simulations to support our theorical result using matlab software version R2014b.

2 Generalities on population evolution models of three species interacting in a given environment

2.1 Reminder of some important notions

A circulant matrix $\mathcal{M} \in \mathbf{M}(\mathbb{C}_n)$ is a square matrix of the following form (1), [11]. In fact, a circulant matrix is a square matrix formed by circular permutations of the coefficients that compose the matrix. In general, in a circulant matrix, you can move from one row to another by moving the coefficients from left to right.

$$\mathcal{M} = \begin{pmatrix} a_0 & a_1 & a_2 & \dots & a_{n-1} \\ a_{n-1} & a_0 & a_1 & \dots & a_{n-2} \\ \vdots & \vdots & \vdots & \dots & \vdots \\ a_1 & a_2 & a_3 & \dots & a_0 \end{pmatrix}, \qquad (1)$$

and that the eigenvalues of such matrix are,

$$\mu_k = \sum_{j=0}^{n-1} a_j \lambda^{jk}, \;\; k = 0, \dots, n-1, \qquad (2)$$

where $\lambda = exp(2\pi i/n)$ and the eigenvectors are,

$$\varpi_k = (1, \lambda^k, \lambda^{2k}, \dots, \lambda^{(n-1)k}), \;\; k = 0, \dots, n-1. \qquad (3)$$

We use the following notations in our work.

$$\mathbb{R}^3_+ = \Big\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 / x_i \geqslant 0, \, i = 1, \dots, 3 \Big\}, \qquad (4)$$

and the interior of this set is denoted as follows

$$Int(\mathbb{R}^3_+) = \Big\{ (x_1, x_2, x_3)^T \in \mathbb{R}^3 / x_i > 0, \, i = 1, \dots, 3 \Big\}. \qquad (5)$$

We recall the following comparison lemma.

Lemma 2.1 *[12] If $A > 0$, $B > 0$ and $\frac{dx}{dt} \geq x(A - Bx)$, when $t \geq 0$ and $x(0) > 0$, we have*

$$\liminf_{t \longrightarrow +\infty} x(t) \geq \frac{A}{B}. \qquad (6)$$

If $A > 0$, $B > 0$ and $\frac{dx}{dt} \leq x(A - Bx)$, when $t \geq 0$ and $x(0) > 0$, we have

$$\limsup_{t \longrightarrow +\infty} x(t) \leq \frac{A}{B}. \qquad (7)$$

Let $n, p \in \mathbb{N}^*$ and consider the following system of autonomous differential equations

$$\begin{cases} \frac{dx}{dt} = f(x), \\ x(0) = x_0, \end{cases} \qquad (8)$$

such that $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^p$ is a map of a non-empty open set Ω of $\mathbb{R}^n$ differentiable at $a \in \Omega$.

Definition 2.1 *[13] The Jacobian matrix of f in a is denoted by*

$$Jac(f)(a) \quad = \quad \left(\frac{\partial f_i}{\partial x_j}(a) \right)_{\substack{1 \leqslant i \leqslant p \\ 1 \leqslant j \leqslant n}} \tag{9}$$

that is the matrix of partial derivatives $\frac{\partial f_i}{\partial x_j}$ *in a, with* $1 \leqslant i \leqslant p$ *and* $1 \leqslant j \leqslant n$*, relative to the canonical bases* $\mathcal{B}_n$ *and* $\mathcal{B}_p$ *of* $\mathbb{R}^n$ *and* $\mathbb{R}^p$ *respectively.*

Lemma 2.2 *[13], [14] Let us consider* $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ *with* $n \in \mathbb{N}^*$ *and let* $a \in \Omega$*. We have*

- *If all the eigenvalues of* $Jac(f)(a)$ *have a strictly negative real part, then the nonlinear system (8) is asymptotically stable.*
- *If at least one eigenvalue has a real part strictly greater than 0, then the nonlinear system (8) is unstable.*
- *If the Jacobian matrix* $Jac(f)(a)$ *has zero eigenvalue or a pair of purely imaginary complex-conjugate eigenvalues, then we are in a critical case; we cannot conclude anything.*

Stable equilibrium points that are not asymptotically stable can only occur at non-hyperbolic equilibrium points. However, to find out whether a non-hyperbolic equilibrium point is stable, asymptotically stable or unstable we resort to Lyapunov's method which is very useful in answering this question.

Theorem 2.1 ([14], [15], [16]) *Consider the following polynomial:*

$$P(X) = a_0X^n + a_1X^{n-1} + a_2X^{n-2} + \cdots + a_{n-1}X + a_n, \tag{10}$$

with $a_i \in \mathbb{R},\ i = \{0, ..., n\}$.
The roots of the polynomial P have all negative real parts if and only if all the principal minors of the Hurwitz matrix are strictly positive.

That is to say, all solutions of the equation $P(X) = 0$, have negative real parts if and only if, the below n principal minors are all positive.

$$\Delta_i = \begin{vmatrix} a_1 & a_0 & 0 & \cdots & 0 \\ a_3 & a_2 & a_1 & & \vdots \\ a_5 & a_4 & a_3 & \ddots & \\ \vdots & \vdots & \vdots & \ddots & a_{i-1} \\ \cdots & \cdots & \cdots & \cdots & a_i \end{vmatrix}, \tag{11}$$

with $(i = 1, ..., n)$.

Theorem 2.2 *[17] Consider* $f : \Omega \subset \mathbb{R}^n \to \mathbb{R}^n$ *and* $V : \mathcal{V} \subset \mathbb{R}^n \to \mathbb{R}^+$ *functions of class* $\mathcal{C}^1$*. Suppose* $\dot{V}(u) \leq 0$ *for all* $u \in \mathcal{V}$*. Let* $\mathcal{I}$ *be a compact, positively invariant subset of* $\mathcal{V}$*. Define B as the set of points in* $\mathcal{I}$ *on which* $\dot{V}(u) = 0$*, i.e.* $B = \left\{u \in \mathcal{I},\ \dot{V}(u) = 0\right\}$ *and* Ω *the largest invariant subset of B.*

Then, all solutions of (8) bounded for $t \geq 0$*, converge to* Ω *when* $t \longrightarrow \infty$*.*

2.2 Model of three competitors of May-Leonard

Scientists Robert McCredie May and Warren Leonard introduced a model of three species competing for the same resource, [18]. The study of this model allowed to examine unexpected behaviors of these competitors. This results in rich population dynamics. The above interactions can be summarized in the figure below, (Figure 1). Indeed, we can find several examples of species that compete with resources available in a given environment. Here, we give an example where selected species are in competition in their environment for the resources found there, (Figure 2).

These species of birds feed on plants (leaves, seeds, nectar and sap) and various animals (invertebrates, small animals, dead animal bodies, fish, slugs and snails).

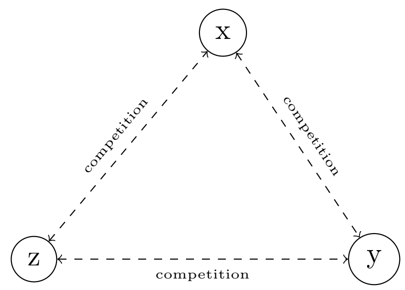


Figure 1: Diagram of competition interactions between three species.

The differential system of the Lotka-Volterra type which describes the competition interaction of these three populations with the same growth rate and density x, y and z is given by the following expression:

$$\begin{cases} \frac{dx}{dt} = x(1 - x - a\,y - b\,z), \\ \frac{dy}{dt} = y(1 - b\,x - y - a\,z), \\ \frac{dz}{dt} = z(1 - a\,x - b\,y - z). \end{cases} \tag{12}$$

Initial densities are $x(0) \geq 0$, $y(0) \geq 0$ and $z(0) \geq 0$. The parameters a and b are such

Figure 2: Example of a food chain in nature. (Images, Planet Animals).

$0 < b < 1$ and $a + b > 2$. In this model, all three species are assumed to have the same intrinsic growth rate, r=1. Parameters a and b represent competition rates, so that the system is cyclically symmetrical. Studying the model (12), we obtain five equilibrium points. The trivial equilibrium points are $(0,0,0)^T$, $(1,0,0)^T$, $(0,0,1)^T$, $(0,1,0)^T$ and the positive equilibrium point is $(x^*, y^*, z^*)^T$ with $x^* = y^* = z^* = \frac{1}{1+a+b}$. The stability of each equilibrium point is examined below. We have the following proposition.

Proposition 2.1 *System (12) of May-Leonard's three competitors has:*

the equilibrium point $(0,0,0)^T$ which is an unstable node.

- *the equilibrium points $(0,0,1)^T$, $(0,1,0)^T$, $(1,0,0)^T$ and $(x^*, y^*, z^*)^T$ which are unstable.*

Proof: The Jacobian matrix of system (12) is given by the following expression:

$$Jac(f) = \begin{pmatrix} a_{11} & -a\,x & -b\,x \\ -b\,y & a_{22} & -a\,y \\ -a\,z & -b\,z & a_{33} \end{pmatrix}. \qquad (13)$$

with

$$\begin{aligned} a_{11} &= 1 - 2\,x - a\,y - b\,z, && (14)\\ a_{22} &= 1 - 2\,y - b\,x - a\,z, && (15)\\ a_{33} &= 1 - 2z - a\,x - b\,y. && (16) \end{aligned}$$

Jacobian matrices at equilibrium points $(0,0,0)^T$, $(1,0,0)^T$, $(0,1,0)^T$, $(0,0,1)^T$ **are**

$$Jac(f)(0,0,0)^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \qquad (17)$$

$$Jac(f)(1,0,0)^T = \begin{pmatrix} -1 & -a & -b \\ 0 & 1-b & 0 \\ 0 & 0 & 1-a \end{pmatrix}, \qquad (18)$$

$$Jac(f)(0,1,0)^T = \begin{pmatrix} 1-a & 0 & 0 \\ -b & -1 & -a \\ 0 & 0 & 1-b \end{pmatrix}, \qquad (19)$$

$$Jac(f)(0,0,1)^T = \begin{pmatrix} 1-b & 0 & 0 \\ 0 & 1-a & 0 \\ -a & -b & -1 \end{pmatrix}, \qquad (20)$$

the eigenvalue of $Jac(f)(0,0,0)^T$ **is 1 and the others Jacobian matrices have the same eigenvalues that are** -1, $1-a$ *and* $1-b$**. Knowing that** $1 - b > 0$**, all equilibrium points are saddle points and therefore unstable. Moreover, the Jacobian matrix at equilibrium point** $(x^*, y^*, z^*)^T$ **is**

$$Jac(f)(x^*, y^*, z^*)^T = \frac{1}{1+a+b} \begin{pmatrix} -1 & -a & -b \\ -b & -1 & -a \\ -a & -b & -1 \end{pmatrix}. \qquad (21)$$

It's a circulant matrix of order 3. Thus using the properties of circulant matrix, the following three eigenvalues are obtained: $\lambda_0 = -1$ **and** $\lambda_1 = \bar{\lambda}_2 = \frac{1}{1+a+b}\big(-1 - a\,exp(\frac{2\pi i}{3}) - b\,exp(\frac{4\pi i}{3})\big)$ **with real part** $\frac{1}{1+a+b}\big(-1 + \frac{a+b}{2}\big)$ **strictly positive. As a result, the equilibrium point** $(x^*, y^*, z^*)^T$ **is a saddle point. So, it's an unstable equilibrium point.** ■

The different variations in population size over time are summarized in the chronicle (Figure 3) below, which illustrates the different evolutions of populations subjected to such interactions in their environment. For the following parameters $\{a = 1.9; b = 0.7\}$ and initial densities $(x_0 = 5; y_0 = 4; z_0 = 3)$, we obtain the graph on which we can see the different density variations of each competitor over time. The behavior of the three competitors is described as follows: for the selected values, we find that for a certain period species x seem to be the only survivor. Then, suddenly, the density of species y dominates that of species x. After a while, it is species y that gives way to species z, which will also give way to species x. And the cycle continues.

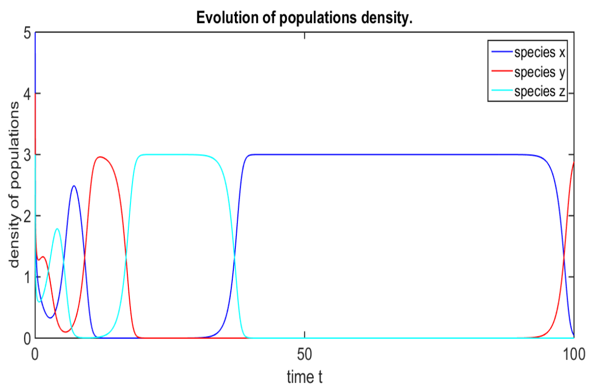


Figure 3: Growth model of May-Leonard's three competitors.

2.3 Prey-predator models of three species

2.3.1 Two preys for one predator

Consider a model consisting of two populations of preys, one of density x and the other of density y. The third population of density z is a predator that exerts its predation on the two preys populations. These preys are competing for the same resource, [13], [19]. We can find several examples of species with a diversified food source. In the given example below, we have rabbit and damson which are herbivores and feed mainly on grasses, leaves, young shoots and fruits they find in their environment. Also, we have Leptailurus-serval, a common predator to both preys.
The above described interactions can be graphically summarized as follows, (Figure 5) on which we can see that the species of density z hunts two competing populations of densities x and y. The system of differential equations of the Lotka-Volterra type that describes the competition and predation interactions between these three species can be described by the following expression:

Figure 4: Example of a food chain in nature. (Images, Planet Animals).

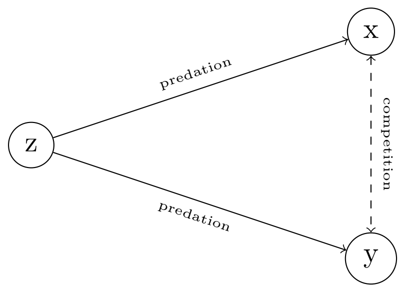


Figure 5: Diagram of interactions between one predator and two preys.

$$\begin{cases} \frac{dx}{dt} = x(1 - x - y - a\,z), \\ \frac{dy}{dt} = y(1 - b\,x - y - z), \\ \frac{dz}{dt} = z(-1 + c_1\,x + c_2\,y - c_3\,z). \end{cases} \quad (22)$$

The parameters a, b, c_1, c_2 and c_3 are all strictly positive and initial population densities are all positive. In this model, a is the predation rate of the predator z on the prey x and b is the competition rate between the two preys x and y. Furthermore, c_1 and c_2 are the assimilation coefficients of preys x and y by the predator z respectively. These are the essential biomasses that the predators use to increase their physical condition and population density. Finally, c_3 is the rate of competition within the predator population. Studying the model (22), we get four equilibrium points: $(0,0,0)^T$, $(1,0,0)^T$, $(0,1,0)^T$ and the positive equilibrium point $E = (x^*, y^*, z^*)^T$ with

$$\begin{aligned} x^* &= \frac{c_2 + a - 1 - ac_2}{c_2 + c_3 + ac_1 - c_1 - abc_2 - bc_3}, \quad (23) \\ y^* &= \frac{1 + ab + abc_2 - c_2}{c_2 + c_3 + ac_1 - c_1 - abc_2 - bc_3}, \quad (24) \\ z^* &= \frac{c_2 + b - bc_2 - 1}{c_2 + c_3 + ac_1 - c_1 - abc_2 - bc_3}. \quad (25) \end{aligned}$$

These components are subject to the following positive conditions.

$$\begin{cases} c_2 + a > 1 + ac_2 \\ 1 + ab + abc_2 > c_2 \\ c_2 + b > 1 + bc_2 \\ c_2 + c_3 + ac_1 > c_1 + abc_2 + bc_3 \end{cases} \quad (26)$$

or

$$\begin{cases} c_2 + a < 1 + ac_2, \\ 1 + ab + abc_2 < c_2, \\ c_2 + b < 1 + bc_2, \\ c_2 + c_3 + ac_1 < c_1 + abc_2 + bc_3. \end{cases} \quad (27)$$

Note that parameters a and b are chosen so that $z^* > 0$.
Let's put $N_1 = 2x^* + y^* + az^*$, $N_2 = 2y^* + bx^* + z^*$ and

$$\begin{aligned} \pi_1 &= (2+b)\,x^* + 3y^* + (1 + a + 2c_3)\,z^*, \\ \pi_2 &= 1 + c_1\,x^* + c_2\,y^*, \\ \pi_3 &= 2(1 + N_1N_2)(1 + 2c_3z^*) + 2bx^*y^* \\ &+ (1 + 2c_3z^* + N_2)(a_{11}^2 + c_2y^*z^*) \\ &+ (1 + 2c_3z^* + N_1)(a_{22}^2 + ac_1x^*z^*) \\ &+ 2(N_1 + N_2)(c_1x^* + c_2z^*) \\ &+ (N_1 + N_2)a_{33}^2 + (c_1 + abc_2)\,x^*y^*z^*, \end{aligned}$$

$$\begin{aligned} \pi_4 &= 2(N_1 + N_2)(1 + 2c_3z^*) + 2a_{33}^2 \\ &+ 2(1 + N_1N_2)(c_1x^* + c_2z^*) + (N_1 + N_2)bx^*y^* \\ &+ (1 + c_1x^* + c_2z^*)(a_{22}^2 + ac_1x^*z^*) \\ &+ (1 + c_1x^* + c_2z^*)(a_{11}^2 + c_2y^*z^*), \\ \pi_5 &= ac_1N_2 + c_2N_1y^*z^* + (1 + N_1N_2)(1 + 2c_2z^*) \\ &+ (N_1 + N_2 + bx^*y^*)(c_1x^* + c_2y^*), \end{aligned}$$

$$\begin{aligned} \pi_6 &= (N_1 + N_2 + bx^*y^*)(1 + 2c_2z^*) \\ &+ (1 + N_1N_2)(c_1x^* + c_2y^*) \\ &+ ac_1 + c_2y^*z^* + (c_1 + abc_2)x^*y^*z^*. \end{aligned}$$

Thus, we have the following proposition

Proposition 2.2 *System (22) of two preys and one predator has:*

- *the equilibrium point $(0,0,0)^T$ which is an unstable saddle point.*
- *the equilibrium point $(1,0,0)^T$ which is stable if $c_1 < 1 < b$ and unstable otherwise.*
- *the equilibrium point $(0,1,0)^T$ which is stable, if $c_2 < 1$ and unstable otherwise.*
- *the equilibrium point $(x^*, y^*, z^*)^T$ which is locally asymptotically stable if following inequalities are satisfied: $\pi_1 > \pi_2$, $\pi_3 > \pi_4$, $\pi_5 > \pi_6$.*

Proof: The Jacobian matrix of the system (22) is given by the following expression:

$$Jac(f)(E) = \begin{pmatrix} A_{11} & -x^* & -a\,x^* \\ -b\,y^* & A_{22} & -y^* \\ c_1\,z^* & c_2\,z^* & A_{33} \end{pmatrix}. \quad (28)$$

with

$$\begin{aligned} A_{11} &= 1 - 2\,x^* - y^* - a\,z^*, \quad (29) \\ A_{22} &= 1 - 2\,y^* - b\,x^* - z^*, \quad (30) \\ A_{33} &= -1 + c_1\,x^* + c_2\,y^* - 2c_3z^*. \quad (31) \end{aligned}$$

The Jacobian matrix at equilibrium point $(0,0,0)^T$ is

$$Jac(f)(0,0,0)^T = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \quad (32)$$

and its eigenvalues are 1 and -1. Then, $(0,0,0)^T$ is an unstable equilibrium point.

Also, the Jacobian matrix at equilibrium point $(1,0,0)^T$ is

$$Jac(f)(1,0,0)^T = \begin{pmatrix} -1 & -1 & -a \\ 0 & 1-b & 0 \\ 0 & 0 & -1+c_1 \end{pmatrix}, \tag{33}$$

and its eigenvalues are -1, 1 - b and $-1+c_1$. If $c_1 < 1 < b$ then the equilibrium point $(1,0,0)^T$ is stable. Otherwise, $(1,0,0)^T$ is an unstable equilibrium point. The Jacobian matrix at equilibrium point $(0,1,0)^T$ is

$$Jac(f)(0,1,0)^T = \begin{pmatrix} 0 & 0 & 0 \\ -b & -1 & -1 \\ 0 & 0 & c_2-1 \end{pmatrix}, \tag{34}$$

and its eigenvalues are 0, -1 and $c_2 - 1$. Then, if $c_2 < 1$, the equilibrium point $(0,1,0)^T$ is a stable equilibrium point. Otherwise, $(0,1,0)^T$ is an unstable equilibrium point.

In addition, we verify that equilibrium point $E = (x^*, y^*, z^*)^T$ is locally asymptotically stable under certain conditions using Routh-Hurwitz criterion. Let's rewrite the Jacobian matrix at equilibrium point $E = (x^*, y^*, z^*)^T$ in the following form.

$$Jac(f)(x^*,y^*,z^*)^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \tag{35}$$

the characteristic polynomial of $Jac(f)(x^*,y^*,z^*)^T$ is

$$P(X) = X^3 + \lambda_2 X^2 + \lambda_1 X + \lambda_0, \tag{36}$$

with

$$\begin{aligned} \lambda_2 &= -(a_{11}+a_{22}+a_{33}), \\ \lambda_1 &= a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33} \\ &- a_{12}a_{21}-a_{23}a_{32}-a_{13}a_{31}, \\ \lambda_0 &= a_{13}a_{31}a_{22}+a_{23}a_{32}a_{11}+a_{12}a_{21}a_{33} \\ &- a_{11}a_{22}a_{33}-a_{12}a_{23}a_{31}-a_{13}a_{21}a_{32}, \end{aligned}$$

according to the Routh-Hurwitz criterion, $(x^*,y^*,z^*)^T$ is stable if $\lambda_2 > 0$, $\lambda_2\lambda_1 > \lambda_0$ and $\lambda_0 > 0$. We put $N_1 = 2x^* + y^* + az^*$ and $N_2 = 2y^* + bx^* + z^*$. By calculation, we obtain the conditions below.

$$\lambda_2 > 0 \quad \Rightarrow$$

$$(2+b)\,x^* + 3y^* + (1+a+2c_3)\,z^* > 1 + c_1\,x^* + c_2\,y^*$$

then,

$$\Rightarrow \quad \pi_1 > \pi_2,$$

moreover,

$$\begin{aligned} \lambda_2\lambda_1 - \lambda_0 &= 2(1+N_1N_2)(1+2c_3z^*) \\ &+ (1+2c_3z^*+N_2)(a_{11}^2+c_2y^*z^*) \\ &+ (1+2c_3z^*+N_1)(a_{22}^2+ac_1x^*z^*) \\ &+ 2(N_1+N_2)(c_1x^*+c_2z^*)+2bx^*y^* \\ &+ (N_1+N_2)a_{33}^2+(c_1+abc_2)\,x^*y^*z^* \\ &- \Big(2(N_1+N_2)(1+2c_3z^*)+2a_{33}^2 \\ &+ 2(1+N_1N_2)(c_1x^*+c_2z^*) \\ &+ (1+c_1x^*+c_2z^*)(a_{22}^2+ac_1x^*z^*) \\ &+ (1+c_1x^*+c_2z^*)(a_{11}^2+c_2y^*z^*) \\ &+ (N_1+N_2)bx^*y^*\Big), \end{aligned}$$

then $\lambda_2\lambda_1 > \lambda_0$ if $\pi_3 > \pi_4$. In addition we have following condition, $\lambda_0 > 0$ if

$$\begin{aligned} & (1+N_1N_2)(1+2c_2z^*) \\ + & (N_1+N_2+bx^*y^*)(c_1x^*+c_2y^*) \\ + & ac_1N_2+c_2N_1y^*z^* \\ > & \Big((N_1+N_2+bx^*y^*)(1+2c_2z^*) \\ + & (1+N_1N_2)(c_1x^*+c_2y^*) \\ + & ac_1+c_2y^*z^*+(c_1+abc_2)x^*y^*z^*\Big). \end{aligned}$$

that is to say $\pi_5 > \pi_6$. Then, the equilibrium point $(x^*,y^*,z^*)^T$ is asymptotically stable if the above conditions are satisfied. ■

The following chronicle (Figure 6), summarizes variations in populations size over the time. We choose following data for the model $\{a = 0.4; b = 0.5; c_1 = 0.2; c_2 = 1.5; c_3 = 0.5\}$ with the initial densities $(x_0 = 2; y_0 = 3; z_0 = 3)$, we obtain the graph on which we can see the different variations in the density of each population over time. The behavior of the three species is described as follows: First, the densities of all three populations decline over a period of time. Then, densities fluctuate over a period of time. Finally, they remain stable from $t = 20$. Note that this stability is a function of the model parameters.

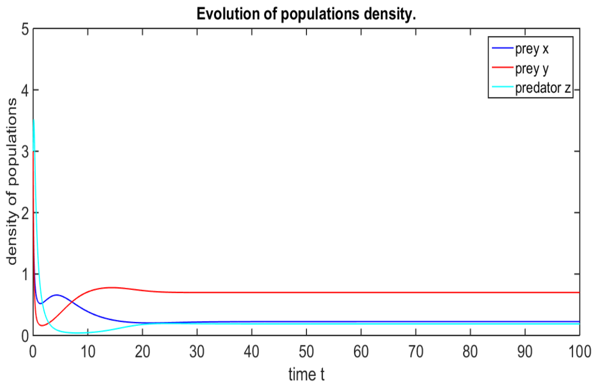


Figure 6: Growth model of two prey for one predator.

2.3.2 Super-predator, predator and prey model

Consider a model consisting of a prey population of density x, a food resource for a predator population of density y, which is also a food resource for a super predator population of density z, so that we have a food chain also called a three-level trophic chain, [13]. In the aquatic environment, for instance, the great white shark commonly preys on large fish like tuna, as well as various mammals and ceteceans. Tuna can feed on small fish such as scomber and many others (mackerel, sardines). The above interactions can be summarized by figure below, (Figure 8).

The system of Lotka-Volterra type differential equations that describes competition and predation interactions between these three species can be described by the following expression

$$\begin{cases} \frac{dx}{dT} = x(r_1 - ay), \\ \frac{dy}{dT} = y(-r_2 + b\,x - cz), \\ \frac{dz}{dT} = r_3 z(1 - \frac{z}{K}) + d\,yz. \end{cases} \quad (37)$$

The parameters are all strictly positive, as the initial population densities. The super predator has a logistic growth with a limiting capacity of the environment noted K.

In this model, for any $j \in \{1, ..., 3\}$, r_j is the growth rate of the index species. In addition, a and c are the predation rates of the predator on the prey respectively of the super predator on the predator. Then, b is the assimilation coefficient of the prey x by the predator y and d that of the predator y by the super predator z. To reduce the number of parameters, we use the following


Figure 7: Example of a food chain in nature. (Images, Monde Animals).

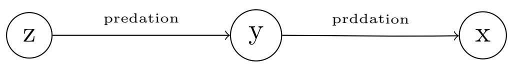


Figure 8: Diagram of interactions between the three species, food chain.

variable changes:

$$t = r_3 T, \quad u = x, \quad v = y, \quad w = \frac{z}{K},$$

$$\alpha_1 = \frac{r_1}{r_3}, \quad \alpha_2 = \frac{a}{r_3}, \quad \beta_3 = \frac{r_2}{r_3}, \quad \beta_1 = \frac{r_2}{r_3},$$

$$\beta_2 = \frac{b}{r_3}, \quad \beta_3 = \frac{cK}{r_3}, \quad \gamma = \frac{d}{r_3},$$

then, the system (37) takes the following form

$$\begin{cases} \frac{du}{dt} = u(\alpha_1 - \alpha_2 v), \\ \frac{dv}{dt} = v(-\beta_1 + \beta_2 u - \beta_3 w), \\ \frac{dw}{dt} = w(1 - w + \gamma v). \end{cases} \quad (38)$$

Studying the model (38), we get two equilibrium points: the trivial equilibrium point $(0, 0, 0)^T$ and the positive equilibrium point $(u^*, v^*, w^*)^T$ such that

$$u^* = \frac{\beta_1}{\beta_2} + \frac{\beta_3}{\beta_2}\big(1 + \frac{\gamma\alpha_1}{\alpha_2}\big), \quad v^* = \frac{\alpha_1}{\alpha_2}$$

and $w^* = 1 + \frac{\gamma\alpha_1}{\alpha_2}$.

Proposition 2.3 *System (38) of prey, predator and super predator has:*

- *the trivial equilibrium point* $(0,0,0)^T$ *which is a unstable saddle point.*
- *the equilibrium point* $(u^*, v^*, w^*)^T$ *which is locally asymptotically stable.*

Proof: The Jacobian matrix of the system (38) is

$$Jac(f)(u,v,w)^T = \begin{pmatrix} a_{11} & -\alpha_2\, u & 0 \\ -\beta_2\, v & a_{22} & -\beta_3\, v \\ 0 & \gamma\, w & a_{33} \end{pmatrix}. \tag{39}$$

$$a_{11} = \alpha_1 - \alpha_2\, v, \tag{40}$$
$$a_{22} = -\beta_1 + \beta_2 u - \beta_3 w, \tag{41}$$
$$a_{33} = 1 - 2\, w + \gamma\, v. \tag{42}$$

(i) The Jacobian matrix of the system (38) at equilibrium point $(0,0,0)^T$ is

$$Jac(f)(0,0,0) = \begin{pmatrix} \alpha_1 & 0 & 0 \\ 0 & -\beta_1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{43}$$

and its eigenvalues are α_1, $-\beta_1$ and 1. Since the eigenvalues α_1 and 1 are positive and the eigenvalue $-\beta_1$ is negative, thus, the equilibrium point $(0,0,0)^T$ is an unstable saddle point.
(ii) The Jacobian matrix of the system (38) at the equilibrium point $(u^*, v^*, w^*)^T$ is

$$Jac(f)(u^*,v^*,w^*) = \begin{pmatrix} 0 & -\alpha_2 u^* & 0 \\ \beta_2\, v^* & 0 & -\beta_3\, v^* \\ 0 & \gamma\, w^* & -w^* \end{pmatrix}, \tag{44}$$

and its characteristic polynomial is given by

$$\begin{aligned} \varrho(\lambda) &= \lambda^3 + w^*\,\lambda^2 + (\alpha_2\beta_2 u^*\, v^* + \alpha_2\,\gamma\, v^*\, w^*)\,\lambda \\ &+ \alpha_2\beta_2\, u^*\, v^*\, w^*, \end{aligned} \tag{45}$$

assuming that

$$a_1 = w^*, \tag{46}$$
$$a_2 = \alpha_2\beta_2 u^*\, v^* + \alpha_2\,\gamma\, v^*\, w^*, \tag{47}$$
$$a_3 = \alpha_2\beta_2\, u^*\, v^*\, w^*, \tag{48}$$

according to Routh Hurwitz criterion, the equilibrium point $(u^*, v^*, w^*)^T$ is locally asymptotically stable if the following inequalities are satisfied.

$$\begin{aligned} a_1 &> 0, \\ a_1\, a_2 - a_3 &> 0, \\ a_3 &> 0. \end{aligned}$$

It is easy to see that a_1 and a_3 are positive and we have

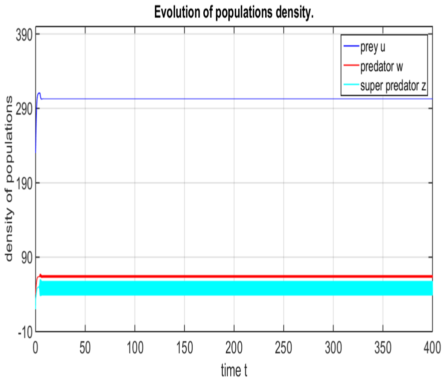
$$\begin{aligned} a_1\, a_2 - a_3 &= \alpha_2\beta_2 u^*\, v^*\, w^* + \alpha_2\,\gamma\, v^*\,(w^*)^2 \\ &- \alpha_2\beta_2\, u^*\, v^*\, w^*, \end{aligned} \tag{49}$$
$$a_1\, a_2 - a_3 = \alpha_2\,\gamma\, v^*\,(w^*)^2 > 0. \tag{50}$$

Then, the Routh Hurwitz's criterion is satisfied. Therefore, the equilibrium point $(u^*, v^*, w^*)^T$ is locally asymptotically stable. ■

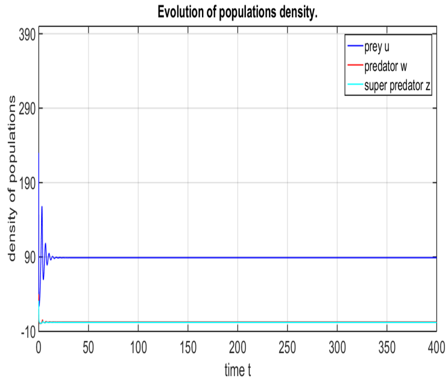
The chronicles (Figure 9), below summarize the variations in population size over time. We choose following data for the model $\{r_1 = 1.09;\ r_2 = 0.71;\ r_3 = 1;\ a = 0.017;\ b = 0.2;\ c = 0.11;\ d = 0.78;\ K = 2\}$ with the initial densities $(u_0 = 230; v_0 = 40; w_0 = 20)$, we obtain the graph Figure 9 on which we can see the different variations in the density of each population over time. The behavior of the three species is described as follows: We can see that the population densities of the prey, predator and superpredator oscillate slightly aperiodically and later converge on a stable state (Figure 9). When the value of the predation rate increases $\{a = 0.41;\ c = 0.5\}$, (Figure 9b) and $\{a = 0.52725;\ c = 0.5\}$, (Figure 9c), we notice a progressive decrease in population densities until extinction.

2.3.3 Prey-predator model with prey harvesting

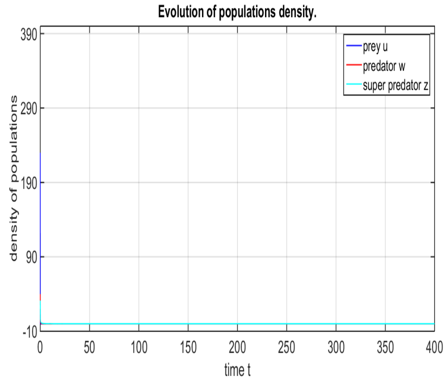
We study a system consisting of a prey population which is a food resource for a natural predator population. Furthermore, it is assumed that the prey population is continuously harvested by harvesting agents, creating competition for prey between harvesters and natural predator, which is not harvested. The authors in [8] have studied the prey-predator model with prey harvesting in order to determine the biomass of the prey stock to be preserved to ensure the regeneration of this resource. Indeed, prey are generally exploited (fished or hunted) for commercial purposes or to satisfy the food needs of the agents that exploit them. However, if these actions go unchecked, prey can disappear very quickly. This could threaten biodiversity.



(a) Population dynamics with $a = 0.017$ and $c = 0.11$



(b) Population dynamics with $a = 0.41$ and $c = 0.5$



(c) Population dynamics with $a = 0.52725$ and $c = 0.5$

Figure 9: Prey, predator and super-predator growth model.

The dynamic model of interactions between prey, predators and harvesting agents, based on a modified Leslie-Gower version and a Holling type II functional response is described by the following expression:

$$\begin{cases} \frac{dx}{dt} = x(r_1 - a\,x - \frac{b\,y}{x+k_1}) - mq\,x\,z, \\ \frac{dy}{dt} = y(r_2 - \frac{c\,y}{x+k_2}), \\ \frac{dz}{dt} = \lambda z(pmqx - d). \end{cases} \tag{51}$$

In this model, the parameters are all strictly positive, the initial population densities are also positive and the super-predator has logistic growth. In addition, x and y are the prey and predator densities. For any $j \in \{1, ..., 2\}$, r_j is the growth rate of the species in question. Moreover, the parameter b represents the predation rate, i.e, the maximum value of the rate of extermination of the prey x by the natural predator y, and c is the maximum value of the rate of extermination of the individual y. In this model, the parameters are all strictly positive, initial population densities are positive, super predator has logistic growth. In addition, x and y are the prey and predator densities and for all $j \in \{1, ..., 2\}$, r_j is the growth rate of the species in question. Moreover, the parameter b represents the predation rate i.e, the maximum value of prey x extermination rate by the natural predator y and c is the maximum value of individual y extermination rate. The reduction in predators population is a consequence of internal competition, natural mortality and mortality due to resource scarcity. Parameter a measures the mortality due to competition between individuals of species x, and parameters k_1 (respectively k_2) measure the protection provided by the environment to prey x (respectively to predator y). Also, the parameter z is the effort used to harvest the prey population and it depends on several factors: first we have m, $0 < m < 1$, which is the fraction of the prey stock available and q which is the prey harvest factor or prey capture factor. The parameter p represents the constant price per unit biomass of prey harvested and d is the cost generated by the constant harvest of prey per unit effort and λ is the rigidity parameter measuring the effort distribution of the reaction.

The above interactions can be summarized in the figure below, (Figure 10).

Studying the model (51), we get five trivial equilibrium points: $(0,0,0)^T$, $(\frac{r_1}{a}, 0, 0)^T$, $(0, \frac{r_2\,k_2}{c}, 0)^T$ and $(x^*, y^*, 0)^T$ which exists if $r_2\sqrt{s} + r_1 r_2 c > r_2^2 b + r_2 ack_1$, with

$$\begin{aligned} x^* &= \frac{r_2\sqrt{s} - r_2^2 b + r_1 r_2 c - r_2 ac(k_1 - 2k_2)}{2r_2 ac} - k_2, \\ y^* &= \frac{r_2\sqrt{s} - r_2^2 b + r_1 r_2 c - r_2 ac(k_1 - 2k_2)}{2ac^2}, \end{aligned} \tag{52}$$

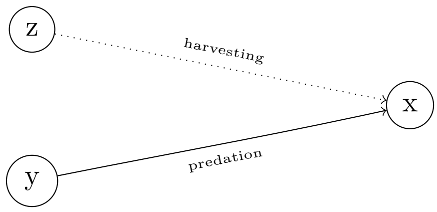


Figure 10: Diagram of interactions between prey, predators and harvesting agents.

Figure 11: Example of a food chain in nature. (Images, Planet Animals).

we have also the equilibrium point $(x^*, 0, z^*)^T$ which exists for $r_1 pmq > ad$, where

$$x^* = \frac{d}{pmq}, \quad y^* = \frac{r_1 pmq - ad}{pm^2 q^2} \tag{53}$$

and positive equilibrium $(x^*, y^*, z^*)^T$ exixts for $r_1 > \frac{ad}{pmq} + \frac{br_2(d+k_2 pmq)}{c(d+k_1 pmq)}$, with

$$\begin{aligned} x^* &= \frac{d}{pmq}, \quad y^* = \frac{r_2}{c}(x^* + k_2) \ \text{and} \\ z^* &= 1 + \frac{1}{mq}\left(\frac{r_1 pmq - ad}{pmq} - \frac{br_2(d + k_2 pmq)}{c(d + k_1 pmq)}\right). \end{aligned}$$

The following proposition gives the conditions for stability of the equilibrium points.

Proposition 2.4 *System (51) of prey-predator model with prey harvesting has:*

- *the trivial equilibrium points* $(0, 0, 0)$ *and* $(x^*, 0, z^*)^T$ *which are all unstable.*
- *the equilibrium point* $(0, \frac{r_2 k_2}{c}, 0)^T$ *which is stable, if* $r_1 < \frac{bk_2 r_2}{ck_1}$.
- *the equilibrium point* $(x^*, y^*, 0)^T$ *which is stable, if* $r_1 < 2a\,x^* + \frac{bk_1 y^*}{(x^*+k_1)^2}$ *and* $pmqx^* < d$.
- *the equilibrium point* $(x^*, y^*, z^*)^T$ *which is locally asymptotically stable if* $r_2 + 2ax^* + \frac{bk_1 y^*}{(x^*+k_1)^2} + mqz^* > r_1 + r_2$.

Proof: We determine the Jacobian matrix of the system (51) at each equilibrium point and examine the sign of its eigenvalues. The Jacobian matrix is given by the following expression.

$$Jac(f)(x, y, z) = \begin{pmatrix} a_{11} & -\frac{bx}{x+k_1} & -mq\,x \\ \frac{cy^2}{(x+k_2)^2} & a_{22} & 0 \\ \lambda pmqz & 0 & a_{33} \end{pmatrix}, \tag{54}$$

with

$$\begin{aligned} a_{11} &= r_1 - 2a\,x - mqz - \frac{bk_1 y}{(x+k_1)^2}, & (55) \\ a_{22} &= r_2 - \frac{2cy}{x+k_2}, & (56) \\ a_{33} &= \lambda(pmqx - d). & (57) \end{aligned}$$

(i) The Jacobian matrix at the equilibrium points $(0, 0, 0)$ and $(\frac{r_1}{a}, 0, 0)$ are respectively given by:

$$Jac(f)(0, 0, 0)^T = \begin{pmatrix} r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & -\lambda d \end{pmatrix}, \tag{58}$$

$$Jac(f)(\frac{r_1}{a}, 0, 0)^T = \begin{pmatrix} -r_1 & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & \lambda(\frac{pmqr_1}{a} - d) \end{pmatrix}. \tag{59}$$

The eigenvalues of $Jac(f)(0, 0, 0)^T$ are r_1, r_2 and $-\lambda d$. Since r_1, r_2 are positive and $-\lambda d$ is negative then, $(0, 0, 0)^T$ is an unstable equilibrium point. Similarly, $(\frac{r_1}{a}, 0, 0)^T$ is an unstable equilibrium point.

(ii) The Jacobian matrix at equilibrium points $(0, \frac{r_2 k_2}{c}, 0)^T$ and $(x^*, y^*, 0)^T$ are respectively given by:

$$Jac(f)(0, \frac{r_2 k_2}{c}, 0)^T = \begin{pmatrix} r_1 - \frac{bk_2 r_2}{ck_1} & 0 & 0 \\ -\frac{r_2^2}{c} & -r_2 & 0 \\ 0 & 0 & -\lambda d \end{pmatrix} \quad (60)$$

and

$$Jac(f)(x^*, y^*, 0)^T = \begin{pmatrix} b_{11} & -\frac{bx^*}{x^*+k_1} & -mq\,x^* \\ \frac{r_2^2}{c} & b_{22} & 0 \\ 0 & 0 & b_{33} \end{pmatrix}, \quad (61)$$

with

$$b_{11} = r_1 - 2a\,x^* - \frac{bk_1 y^*}{(x^* + k_1)^2}, \quad (62)$$
$$b_{22} = -r_2, \quad (63)$$
$$b_{33} = \lambda(pmqx^* - d), \quad (64)$$

The eigenvalues of $Jac(f)(0, \frac{r_2 k_2}{c}, 0)^T$ are $r_1 - \frac{bk_2 r_2}{ck_1}$, $-r_2$ and $-\lambda d$. If $r_1 < \frac{bk_2 r_2}{ck_1}$ then, the equilibrium point $(0, \frac{r_2 k_2}{c}, 0)^T$ is stable. Otherwise it is an unstable equilibrium point.

The equilibrium point $(x^*, y^*, 0)^T$ is unstable if $pmqx^* > d$. Furthermore, let's determine the characteristic polynomial of the Jacobian matrix at the equilibrium point $(x^*, y^*, 0)^T$. We have

$$\varrho(X) = (\lambda(pmqx^* - d) - X)\left(X^2 + \kappa_1 X + \kappa_0\right), \quad (65)$$

with

$$\kappa_1 = r_2 - \left(r_1 - 2a\,x^* - \frac{bk_1 y^*}{(x^* + k_1)^2}\right), \quad (66)$$
$$\kappa_0 = -r_2\left(r_1 - 2a\,x^* - \frac{bk_1 y^*}{(x^* + k_1)^2}\right). \quad (67)$$

The equilibrium point $(x^*, y^*, 0)^T$ is stable if following inequalities are satisfied.

$$\begin{cases} \lambda(pmqx^* - d) < 0 \\ \kappa_1 > 0 \\ \kappa_0 > 0. \end{cases} \quad (68)$$

Solving these equations, we have the following solutions.

$$pmqx^* < d \quad and \quad r_1 < 2a\,x^* + \frac{bk_1 y^*}{(x^* + k_1)^2}. \quad (69)$$

Moreover, the Jacobian matrix at the equilibrium point $(x^*, 0, z^*)^T$ is

$$Jac(f)(x^*, 0, z^*)^T = \begin{pmatrix} B_{11} & B_{12} & -\frac{d}{p} \\ 0 & r_2 & 0 \\ B_{31} & 0 & 0 \end{pmatrix}, \quad (70)$$

with

$$B_{11} = -\frac{ad}{pmq}, \quad (71)$$
$$B_{12} = -\frac{bd}{pmqk_1 + d}, \quad (72)$$
$$B_{31} = \lambda(\frac{r_1 pmq - ad}{mq}). \quad (73)$$

one of its eigenvalues, r_2 is positive, then equilibrium $(x^*, y^*, 0)^T$ is unstable. We check that positive equilibrium $(x^*, y^*, z^*)^T$ is locally asymptotically stable using Routh-Hurwitz criterion. Let's rewrite the Jacobian matrix at the equilibrium point $(x^*, y^*, z^*)^T$ in the following form.

$$Jac(f)(x^*, y^*, z^*)^T = \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix}, \quad (74)$$

with

$$a_{11} = r_1 - 2ax^* - \frac{bk_1 y^*}{(x^* + k_1)^2}, \quad a_{21} = -\frac{r_2^2}{c},$$
$$a_{13} = -mqx^*, \quad a_{12} = \frac{bx^*}{x + k_1},$$
$$a_{22} = -r_2, \quad a_{23} = 0, \quad a_{32} = 0,$$
$$a_{31} = \lambda pmqz^*, \quad a_{33} = \lambda(pmqx^* - d).$$

The characteristic polynomial of $Jac(f)(x^*, y^*, z^*)^T$ is

$$P(X) = X^3 + \rho_2 X^2 + \rho_1 X + \rho_0, \quad (75)$$

with

$$\rho_2 = r_2 - r_1 + 2ax^* + \frac{bk_1 y^*}{(x^* + k_1)^2} + mqz^*,$$
$$\rho_0 = \lambda pm^2 q^2 r_2 x^* z^*,$$
$$\rho_1 = -r_1 r_2 + 2r_2 ax^* + \frac{r_2 bk_1 y^*}{(x^* + k_1)^2} + r_2 mqz^* + \lambda pm^2 q^2 x^* z^* + \frac{r_2^2 bx^*}{c(x^* + k_1)},$$

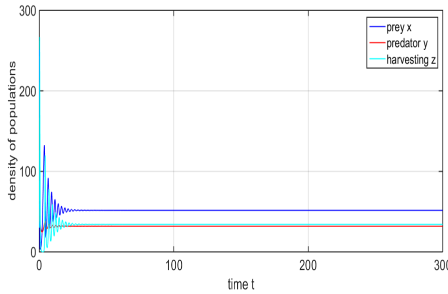
according to the Routh-Hurwitz criterion, $(x^*, y^*, z^*)^T$ is stable if, $\rho_2 >$

$0,\ \rho_2\rho_1 > \rho_0,\ \rho_0 > 0$. Let's put $M{=}r_2 + 2ax^* + \frac{bk_1y^*}{(x^*+k_1)^2} + mqz^*$. A calculation gives the following conditions: $\rho_2 > 0 \Rightarrow M - r_1 > 0$ and $\rho_0 \geq 0$ Furthermore, $\rho_2\rho_1 - \rho_0 > 0 \Rightarrow (r_2(M-r_1) + \lambda pm^2q^2x^*z^*)(M - r_1 - r_2) + \frac{r_2^2bx^*}{c(x^*+k_1)}(M - r_1) > 0$. Finally, the positive equilibrium $(x^*, y^*, z^*)^T$ is locally asymptotically stable if $M > r_1 + r_2$.

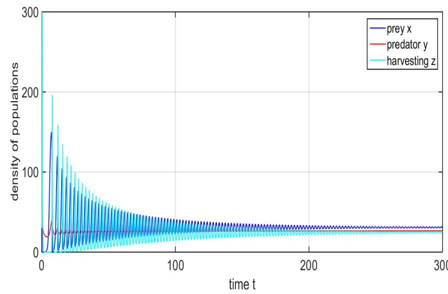
■

The evolution of populations over time is summarized in the following figures, (Figure 12).

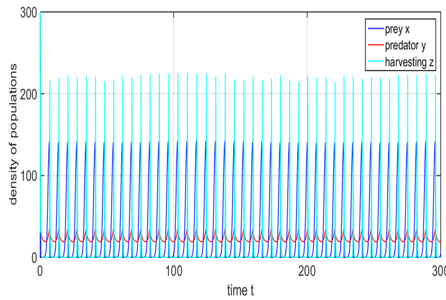
We choose following data for the model $\{r_1 = 2.175;\ r_2 = 0.78;\ a = 0.01375;\ b = 0.16;\ c = 2.98;\ k_1 = 100;\ k_2 = 70;\ \lambda = 11.75;\ d = 0.387;\ p = 0.1789;\ m = 0.016\}$ and we choose different values for q to see the effect of this factor, $\{q = 4.19;\ q = 7.0;\ q = 9.37;\ q = 9.40\}$ with the initial densities $(x_0 = 251; y_0 = 51; z_0 = 30)$. We obtain the graphs (Figure 12), on which we can see the different variations in the density of each population over time. The behavior of the three competitors is described as follows: We observe that prey and predator population densities and harvest rates converge towards their stable states ($x_{stable} \simeq 49;\ y_{stable} \simeq 33;\ z_{stable} \simeq 31$) after a certain period of fluctuation Figure 12a. But, as this stability is conditional, it can also be broken for certain parameter values. In fact, when the factor q increases, the time it takes for the prey to reproduce increases, we can see it on the graphs Figure 12b and Figure 12c. So, when q reaches 9.40, the prey becomes extinct and so does the system. That is shown by Figure 12d.



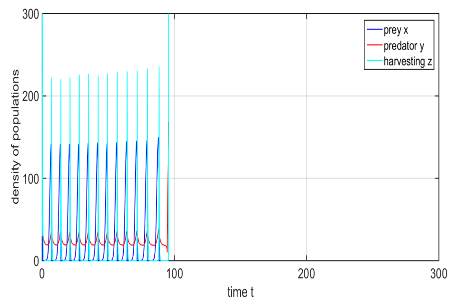
(a) Population dynamics with $q = 4.19$



(b) Population dynamics with q=7.0



(c) Population dynamics with $q = 9.37$



(d) Population dynamics with $q = 9.4$

Figure 12: Prey-predator growth model with prey harvesting.

3 Prey-predator and super predator model including competition, self defense and infected preys

3.1 Presentation of the model

We are interested in the demographic dynamics of prey and predators in a system were Some prey defend themselves vigorously against predators, often causing enormous damage to the predator population. Also the consumption of certain infected prey can lead to disease in the predator population, resulting in high reduction rates. A mathematical model following the assumptions and interactions described above is represented by the diagram (Figure 13) below.

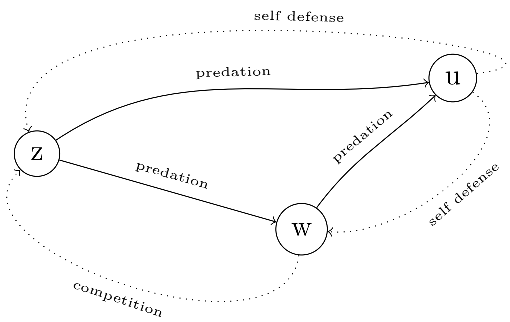


Figure 13: Schematic of prey-predator interactions.

There are examples in nature, Figure 14 illustrating this type of trophic allowing us to describe these prey-predator relationships.

$$\begin{cases} \dfrac{du}{dt} &= r_1 u\left(1-\dfrac{u}{k}\right) - a_1 uw - a_2 uz - \tau u, \\ \dfrac{dw}{dt} &= r_2 w\left(1-\dfrac{w}{k}\right) + b_1 uw - b_2 wz - \alpha uw, \\ \dfrac{dz}{dt} &= r_3 z\left(1-\dfrac{z}{k}\right) + c_1 uz + c_2 wz - \beta uz - \gamma wz. \end{cases} \tag{76}$$

The parameter r_i for all $i = 1,\ 2$ is the growth rate of the corresponding species. The initial densities are all positive. In this model, a_i, for all $i = 1,\ 2$ and b_2 are the predation rates of the predators and b_1, c_1 and c_2 are the coefficients of assimilation of the preys by the predators. k is the carying capacity and τ is infected prey rate. α, β and γ are predator reduction rate due to prey defense in group and consumption of infected prey.

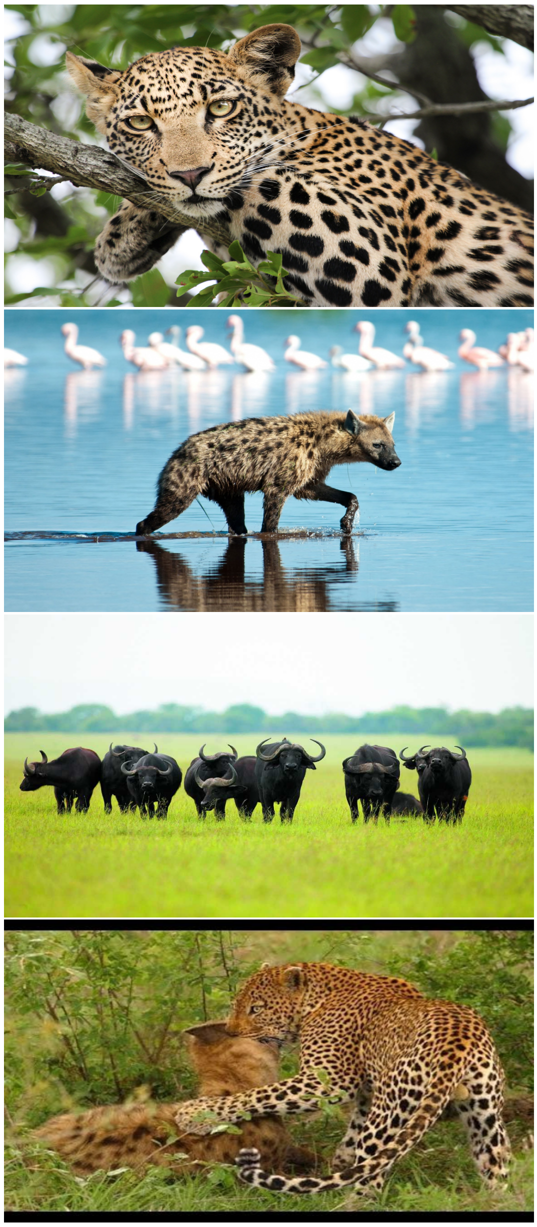


Figure 14: Example of a food chain in nature. (Images, Planet Animals).

The parameters α, β and γ can generally be a factor in increasing the mortality rate of predators, i.e. all the possible negative effects that the prey can exert on its predator. Whether it's self-defense in a group, strategies to escape predators (indirectly inflicting a loss of energy, time and strength), or negative effects due to the consumption of infected prey.

3.2 Boundaries conditions of solutions of system (76)

Definition 3.1 *An equilibrium point* $E^* = (u^*, w^*, z^*)^T$ *of system (76) is said to be non-trivial or interior or again positive, if it belongs to the strictly positive cone* $Int(\mathbb{R}^3_+)$.

Definition 3.2 *[19] A solution of system (76) is said to be ultimately bounded with respect to* $\mathbb{R}^3_+$, *if there exists a compact region* $\mathcal{A}$ *of* $\mathbb{R}^3_+$ *and a finite time* T *such that, for any initial condition* $(t_0, u_0, w_0, z_0)^T \in \mathbb{R}_+ \times \mathbb{R}^3_+$, *we have* $(u, w, z)^T \in \mathcal{A}$ *for all* $t > T$.

Definition 3.3 *[20] The subset* Ω *of* $\mathbb{R}^3_+$ *is a positively invariant region for the solutions of the problem (76), if any solution* $(u, w, z)^T$ *whose initial condition* $(u_0, w_0, z_0)^T$ *is in* Ω *satisfies*

$$(u, w, z)^T \in \Omega, \quad \forall t > 0. \tag{77}$$

Theorem 3.1 *Consider the set* Ω *defined by*

$$\Omega = \left\{ \begin{array}{c} (u, w, z)^T \in \mathbb{R}^3_+ \ \ \text{such that} \\ 0 \le u \le \delta_1, \\ 0 \le w \le \delta_2, \\ 0 \le z \le \delta_3, \end{array} \right\} \tag{78}$$

where

$$\begin{aligned} \delta_1 &= k, \ \ \delta_2 = k\frac{(r_2 + b_1\delta_1)}{r_2} \ \ \text{and} \\ \delta_3 &= \frac{k(r_3 + c_1\,\delta_1 + c_2\,\delta_2)}{r_3}. \end{aligned}$$

Then,

i) The set Ω *is a positively invariant region.*

ii) All solutions of (76) with initial conditions in Ω, *are bounded and enter the attraction region* Ω.

iii) All solutions of system (76) with positive initial conditions are uniformly bounded.

Proof:

Let us prove for each equation of the system (76).

From the first equation of system (76), **we have**

$$\begin{cases} \dfrac{du}{dt} = r_1 u \left(1 - \dfrac{u}{k}\right) - a_1 uw - a_2 uz - \tau u, \\ u(0) = u_{01} \ge 0, \end{cases} \tag{79}$$

and with $-(a_1 uw + a_2 uz + Iu) \le 0$, **we have**

$$\begin{cases} \dfrac{du}{dt} \le r_1 u \left(1 - \dfrac{u}{k}\right) \\ u(0) = u_{01} \ge 0 \end{cases} \tag{80}$$

Then, according to Lemma 2.1,

$$\limsup_{t \longrightarrow \infty} u(t) \le k (\approx \delta_1). \tag{81}$$

Therefore, $\forall \epsilon_1 > 0$, **there exists** $T > 0$ **such that**

$$u(t) \le \delta_1 + \epsilon_1, \quad \forall\, t \ge T. \tag{82}$$

Likewise, considering the second equation of system (76), **we have**

$$\begin{cases} \dfrac{dw}{dt} = r_2 w \left(1 - \dfrac{w}{k}\right) + b_1 uw - b_2 wz - \alpha uw, \\ w(0) = w_{02} \ge 0, \end{cases} \tag{83}$$

and with $-b_2 wz - \alpha uw \le 0$, **we have**

$$\begin{cases} \dfrac{dw}{dt} \le r_2 w \left(1 - \dfrac{w}{k}\right) + b_1 uw, \\ w(0) = w_{02} \ge 0, \end{cases}$$

$$\Longrightarrow \begin{cases} \frac{dw}{dt} \le \left(r_2 + b_1(\delta_1 + \epsilon_1) - \frac{r_2 w}{k}\right) w, \\ w(0) = w_{02} \ge 0, \end{cases} \tag{84}$$

according to Lemma 2.1

$$\limsup_{t \longrightarrow +\infty} w(t) \le \frac{k}{r_2}\Big(r_2 + b_1(\delta_1 + \epsilon)\Big)(\approx \delta_2). \tag{85}$$

Therefore, $\forall \epsilon_2 > 0$, **there exists** $T > 0$ **such that**

$$w(t) \le \delta_2 + \epsilon_2 \ \ \forall t \ge T. \tag{86}$$

We now consider the third equation of system (76), **we have**

$$\begin{cases} \dfrac{dz}{dt} = r_3 z \left(1 - \dfrac{z}{k}\right) + c_1 uz + c_2 wz - \beta uz - \gamma wz, \\ z(0) = z_{03} \ge 0, \end{cases} \tag{87}$$

as $-\beta uz - \gamma wz \le 0$ **and taking** $\varepsilon = max\,\{\epsilon_1, \epsilon_2\}$, **we have** $\forall t \ge T > 0$,

$$u(t) \le \delta_1 + \varepsilon \quad \text{and} \quad w(t) \le \delta_2 + \varepsilon,$$

then we get

$$\begin{cases} \dfrac{dz}{dt} \leq r_3 z\left(1 - \dfrac{z}{k}\right) + c_1 uz + c_2 wz \\ z(0) = z_{03} \geq 0 \end{cases}$$

$$\Longrightarrow \begin{cases} \frac{dz}{dt} \leq \Big(r_3 + c_1(\delta_1+\varepsilon) + c_2(\delta_2+\varepsilon) - \frac{r_3\, z}{k}\Big)z, \\ z(0) = z_{03} \geq 0. \end{cases} \quad (88)$$

According to Lemma 2.1

$$\limsup_{t\longrightarrow+\infty} z(t) \leq \frac{k}{r_3}\Big(r_3 + c_1(\delta_1+\varepsilon) + c_2(\delta_2+\varepsilon)\Big), \quad (89)$$

when $\varepsilon \to 0$, it turns out that

$$\limsup_{t\longrightarrow+\infty} z(t) \leq \frac{k}{r_3}\Big(r_3 + c_1\,\delta_1 + c_2\,\delta_2\Big)(\equiv \delta_3). \quad (90)$$

Furthermore, for all $t_0 \geq 0$, take $X(t_0) = \Big(u(0), w(0), z(0)\Big)^T$ with positive components and let $\sigma(t) = u(t) + w(t) + z(t)$. Then, by deriving along the solutions of the system (76) we have $\forall \zeta > 0$

$$\begin{aligned} \frac{d\sigma(t)}{dt} \;&+\; \zeta\sigma(t) \;\leq\; \frac{k}{4}\left(3 + \frac{(\zeta-\tau)}{r_1}\right) \\ &+\; \frac{k}{4}\left(\frac{(\zeta + b_1\delta_1)}{r_2} + \frac{(\zeta + c_1\delta_1 + c_2\delta_2)}{r_3}\right), \\ &\approx\; (m), \end{aligned} \quad (91)$$

applying Gronwall's inequality, we obtain

$$0 < \sigma(X(t)) \leq \frac{m}{\zeta}\Big(1 - e^{-\zeta t}\Big) + \sigma(X(t_0))e^{-\zeta t}, \quad (92)$$

then when $t \longrightarrow +\infty$ we have

$$0 < \sigma(X(t)) \leq \frac{m}{\zeta}. \quad (93)$$

The solutions of system (76) are uniformly bounded for any initial condition $X(t_0) \geq 0$. ■

3.3 Study of system equilibrium points

The equilibrium points E_j^*, $(j = 0, 1, 2, \dots)$ are the solutions of following equations:

$$\Big\{ \frac{du}{dt} = 0; \quad \frac{dw}{dt} = 0; \quad \frac{dz}{dt} = 0. \quad (94)$$

- The trivial equilibrium point of the system (76) is $E_0^* = (0,0,0)^T$.
- The semi-trivial equilibrium points of the system (76) are $E_1^* = \left(k(1 - \dfrac{\tau}{r_1}), 0, 0\right)^T$, $E_2^* = (0, k, 0)^T$, $E_3^* = (0,0,k)^T$ and $E_4^* = (u^*, w^*, 0)^T$ with

$$\begin{aligned} u^* &= \frac{r_2 k(r_1 - \tau - a_1 k)}{r_1 r_2 + a_1(b_1 - \alpha)}, \\ w^* &= \frac{k(r_1 r_2 + (r_1 - \tau)(b_1 - \alpha))}{r_1 r_2 + k a_1(b_1 - \alpha)}, \end{aligned}$$

and $E_5^* = (u^*, 0, z^*)^T$ with

$$\begin{aligned} u^* &= \frac{r_3 k(r_1 - \tau - a_2 k)}{r_1 r_3 + a_2(c_1 - \beta)}, \\ z^* &= \frac{k\,(r_1 r_3 + (r_1 - \tau)(c_1 - \beta))}{r_1 r_3 + k a_2(c_1 - \beta)}, \end{aligned}$$

and $E_6^* = (0, w^*, z^*)^T$ with

$$\begin{aligned} w^* &= \frac{r_3 k(r_2 - b_2 k)}{r_2 r_3 + k b_2(c_2 - \gamma)}, \\ z^* &= \frac{r_2 k(r_3 + c_2 - \gamma)}{r_2 r_3 + k b_2(c_2 - \gamma)}. \end{aligned}$$

- The positive equilibrium point of system (76) is $\bar{E} = (\bar{u}, \bar{w}, \bar{z})^T$ with

$$\begin{aligned} \bar{u} &= \frac{k(r_1 - a_1 A - a_2 C - \tau)}{r_1 + k a_1 B + k a_2 D}, \\ \bar{w} &= \frac{k r_3(r_2 - k b_2) + k(r_3(b_1 - \alpha)}{r_2 r_3 + k^2 b_2(c_2 - \gamma)} \\ &- \frac{k b_2(c_1 - \beta))\bar{u}}{r_2 r_3 + k^2 b_2(c_2 - \gamma)}, \\ \bar{z} &= k\frac{(r_2 r_3 + 2k^2 b_2 c_2 + k r_2(c_2 - \gamma))}{r_2 r_3 + k b_2(c_2 - \gamma)} \\ &+ k\frac{(2r_2(c_1 - \beta) + k(c_2 - \gamma)(c_1 - \alpha))\bar{u}}{r_2 r_3 + k b_2(c_2 - \gamma)}, \end{aligned}$$

with

$$\begin{aligned} A &= \frac{k r_3(r_2 - k b_2)}{r_2 r_3 + k^2 b_2(c_2 - \gamma)}, \\ B &= \frac{k(r_3(b_1 - \alpha) - k b_2(c_1 - \beta))}{r_2 r_3 + k^2 b_2(c_2 - \gamma)}\bar{u}, \\ C &= k\frac{(r_2 r_3 + 2k^2 b_2 c_2 + k r_2(c_2 - \gamma))}{r_2 r_3 + k b_2(c_2 - \gamma)}, \\ D &= k\frac{(2r_2(c_1 - \beta) + k(c_2 - \gamma)(c_1 - \alpha))}{r_2 r_3 + k b_2(c_2 - \gamma)}\bar{u}. \end{aligned}$$

It's clear that equilibrium points E_2^* and E_3^* exist because their components are positive and equilibrium point E_1^* exists for $r_1 > \tau$.

Lemma 3.1 *The equilibrium point E_4^* exists for the system (76) if the following hypothesis are satisfied.*

$$\begin{cases} r_1 > \tau + a_1 k, \quad r_1 r_2 + k a_1 b_1 > k a_1 \alpha, \\ r_1 r_2 + r_1 b_1 + \tau\alpha > b_1 \tau + r_1 \alpha, \end{cases} \tag{95}$$

or

$$\begin{cases} r_1 < \tau + a_1 k, \quad r_1 r_2 + k a_1 b_1 < k a_1 \alpha, \\ r_1 r_2 + r_1 b_1 + \tau\alpha < b_1 \tau + r_1 \alpha. \end{cases} \tag{96}$$

Proof: As $z^* = 0$ in E_4^*, the system (76) is reduce to the following system

$$\begin{cases} \dfrac{du}{dt} &= r_1 u \left(1 - \dfrac{u}{k}\right) - a_1 u w - \tau u, \\ \dfrac{dw}{dt} &= r_2 w \left(1 - \dfrac{w}{k}\right) + b_1 u w - \alpha u w, \end{cases} \tag{97}$$

and applying (94) we get

$$\begin{cases} 0 &= r_1 - \dfrac{r_1 u}{k} - a_1 w - \tau, \\ 0 &= r_2 - \dfrac{r_2 w}{k} + (b_1 - \alpha) u, \end{cases}$$

$$\Rightarrow \begin{cases} u^* &= k - \dfrac{k a_1 w^*}{r_1} - \dfrac{k\tau}{r_1}, \\ w^* &= k + \dfrac{(b_1 - \alpha)}{r_2} u^*, \end{cases}$$

$$\Rightarrow \begin{cases} u^* &= \dfrac{r_2 k (r_1 - \tau - a_1 k)}{r_1 r_2 + k a_1 (b_1 - \alpha)}, \\ w^* &= k + \dfrac{(b_1 - \alpha)}{r_2} u^*, \end{cases}$$

for biological reasons, E_4^* exists if its components are positive. That lead to $u^* > 0$ and $w^* > 0$ then,

$$\frac{r_2 k (r_1 - \tau - a_1 k)}{r_1 r_2 + k a_1 (b_1 - \alpha)} > 0$$
$$\text{and} \qquad \frac{k(r_1 r_2 + (r_1 - \tau)(b_1 - \alpha))}{r_1 r_2 + k a_1 (b_1 - \alpha)} > 0,$$

it follows

$$\begin{cases} r_1 > \tau + a_1 k, \quad r_1 r_2 + k a_1 b_1 > k a_1 \alpha, \\ r_1 r_2 + r_1 b_1 + \tau\alpha > b_1 \tau + r_1 \alpha, \end{cases} \tag{98}$$

or

$$\begin{cases} r_1 < \tau + a_1 k, \quad r_1 r_2 + k a_1 b_1 < k a_1 \alpha, \\ r_1 r_2 + r_1 b_1 + \tau\alpha < b_1 \tau + r_1 \alpha. \end{cases} \tag{99}$$

■

Lemma 3.2 *The equilibrium point E_5^* exists for the system (76) if the following hypothesis are satisfied.*

$$\begin{cases} r_1 > \tau + a_2 k, \quad r_1 r_3 + k a_2 c_1 > k a_2 \beta, \\ r_1 r_3 + r_1 c_1 + \tau\beta > c_1 \tau + r_1 \beta, \end{cases} \tag{100}$$

or

$$\begin{cases} r_1 < \tau + a_2 k, \quad r_1 r_3 + k a_2 c_1 < k a_2 \beta, \\ r_1 r_3 + r_1 c_1 + I\beta < c_1 \tau + r_1 \beta. \end{cases} \tag{101}$$

Proof: As $w^* = 0$ in E_5^*, the system (76) is reduce to following system

$$\begin{cases} \dfrac{du}{dt} &= r_1 u \left(1 - \dfrac{u}{k}\right) - a_2 u z - \tau u, \\ \dfrac{dz}{dt} &= r_3 z \left(1 - \dfrac{z}{k}\right) + c_1 u z - \beta u z, \end{cases} \tag{102}$$

and applying (94) we get

$$\begin{cases} 0 &= r_1 - \dfrac{r_1 u}{k} - a_2 z - \tau, \\ 0 &= r_3 - \dfrac{r_3 z}{k} + (c_1 - \beta) u, \end{cases}$$

$$\Rightarrow \begin{cases} u^* &= k - \dfrac{k a_2 z^*}{r_1} - \dfrac{k\tau}{r_1}, \\ z^* &= k + \dfrac{(c_1 - \beta)}{r_3} u^*, \end{cases}$$

$$\Rightarrow \begin{cases} u^* &= \dfrac{r_3 k (r_1 - I - a_2 k)}{r_1 r_3 + k a_2 (c_1 - \beta)}, \\ z^* &= k + \dfrac{(c_1 - \beta)}{r_3} u^*, \end{cases}$$

for biological reasons, E_5^* exists if its components are positive. That lead to $u^* > 0$ and $z^* > 0$ then,

$$\frac{r_3 k (r_1 - \tau - a_2 k)}{r_1 r_3 + k a_2 (c_1 - \beta)} > 0$$
$$\text{and} \qquad \frac{k\left(r_1 r_3 + (r_1 - \tau)(c_1 - \beta)\right)}{r_1 r_3 + k a_2 (c_1 - \beta)} > 0,$$

it follows

$$\begin{cases} r_1 > \tau + a_2 k, \quad r_1 r_3 + k a_2 c_1 > k a_2 \beta, \\ r_1 r_3 + r_1 c_1 + \tau\beta > c_1 \tau + r_1 \beta, \end{cases} \tag{103}$$

or

$$\begin{cases} r_1 < \tau + a_2 k, \quad r_1 r_3 + k a_2 c_1 < k a_2 \beta, \\ r_1 r_3 + r_1 c_1 + \tau\beta < c_1 \tau + r_1 \beta. \end{cases} \tag{104}$$

■

Lemma 3.3 *The equilibrium point E_6^* exists for the system (76) if the following hypothesis are satisfied.*

$$\begin{cases} r_2 > b_2 k, \quad r_2 r_3 + k b_2 c_2 > k b_2 \gamma, \quad r_3 + c_2 > \gamma, \\ or \\ r_2 < b_2 k, \quad r_2 r_3 + k b_2 c_2 < k b_2 \gamma, \quad r_3 + c_2 < \gamma. \end{cases} \tag{105}$$

Proof: As $u^* = 0$ in E_6^*, the system (76) is reduce to following system

$$\begin{cases} \dfrac{dw}{dt} = r_2 w \left(1 - \dfrac{w}{k}\right) - b_2 w z, \\ \dfrac{dz}{dt} = r_3 z \left(1 - \dfrac{z}{k}\right) + c_2 w z - \gamma w z, \end{cases} \tag{106}$$

and applying (94) we get

$$\begin{cases} 0 = r_2 - \dfrac{r_2 w}{k} - b_2 z, \\ 0 = r_3 - \dfrac{r_3 z}{k} + (c_2 - \gamma) w, \end{cases}$$

$$\Rightarrow \begin{cases} w^* = k - \dfrac{k b_2 z^*}{r_2}, \\ z^* = k + \dfrac{(c_2 - \gamma)}{r_3} w^*, \end{cases}$$

$$\Rightarrow \begin{cases} w^* = \dfrac{r_3 k (r_2 - b_2 k)}{r_2 r_3 + k b_2 (c_2 - \gamma)}, \\ z^* = \dfrac{r_2 k (r_3 + c_2 - \gamma)}{r_2 r_3 + k b_2 (c_2 - \gamma)}, \end{cases}$$

for biological reasons, E_6^* exists if its components are positive. That lead to $w^* > 0$ and $z^* > 0$ then,

$$\frac{r_3 k (r_2 - b_2 k)}{r_2 r_3 + k b_2 (c_2 - \gamma)} > 0,$$
$$and \qquad \frac{r_2 k (r_3 + c_2 - \gamma)}{r_2 r_3 + k b_2 (c_2 - \gamma)} > 0,$$

it follows

$$\begin{cases} r_2 > b_2 k, \quad r_2 r_3 + k b_2 c_2 > k b_2 \gamma, \quad r_3 + c_2 > \gamma, \\ or \\ r_2 < b_2 k, \quad r_2 r_3 + k b_2 c_2 < k b_2 \gamma, \quad r_3 + c_2 < \gamma. \end{cases} \tag{107}$$

■

Lemma 3.4 *The positive equilibrium point $\bar{E}$ exists for the system (76) if the following hypothesis are satisfied.*

$$r_2 > b_2 k, \quad c_2 > \gamma, \quad b_1 > \alpha, \quad c_1 > \beta,$$
$$r_1 > a_1 A + a_2 C + \tau \;\; and$$
$$r_3 r_2 + r_3 b_1 + b_2 \beta \bar{u} > r_3 k b_2 + r_3 \alpha + b_2 c_1 \bar{u}.$$

Proof: Using the system (76) and applying (94) we get

$$\begin{cases} r_1 \left(1 - \dfrac{u}{k}\right) - a_1 w - a_2 z - \tau = 0, \\ r_2 \left(1 - \dfrac{w}{k}\right) + (b_1 - \alpha) u - b_2 z = 0, \\ r_3 \left(1 - \dfrac{z}{k}\right) + (c_1 - \beta) u + (c_2 - \gamma) w = 0, \end{cases}$$

$$\Rightarrow \begin{cases} u = \dfrac{k}{r_1} \left(r_1 - a_1 w - a_2 z - \tau\right), \\ w = \dfrac{k}{r_2} \left(r_2 + (b_1 - \alpha) u - b_2 z\right), \\ z = \dfrac{k}{r_3} \left(r_3 + (c_1 - \beta) u + (c_2 - \gamma) w\right), \end{cases}$$

and using third and second equation we get

$$\begin{aligned} w &= -\frac{k^2 b_2}{r_2 r_3} \left(r_3 + (c_1 - \beta) u + (c_2 - \gamma) w\right) \\ &+ \frac{k}{r_2} \Big(r_2 + (b_1 - \alpha) u\Big), \\ \Rightarrow \bar{w} &= \frac{k (r_3 (b_1 - \alpha) - k b_2 (c_1 - \beta)) \bar{u}}{r_2 r_3 + k^2 b_2 (c_2 - \gamma)} \\ &+ \frac{k r_3 (r_2 - k b_2)}{r_2 r_3 + k^2 b_2 (c_2 - \gamma)}, \end{aligned}$$

then,

$$\begin{aligned} z &= \frac{k}{r_3} \left(r_3 + (c_1 - \beta) u + (c_2 - \gamma) w\right), \\ \Rightarrow \bar{z} &= k \frac{(r_2 r_3 + 2k^2 b_2 c_2 + k r_2 (c_2 - \gamma))}{r_2 r_3 + k b_2 (c_2 - \gamma)} \\ &+ k \frac{(2 r_2 (c_1 - \beta) + k (c_2 - \gamma)(c_1 - \alpha)) \bar{u}}{r_2 r_3 + k b_2 (c_2 - \gamma)}, \end{aligned}$$

then let

$$\begin{aligned} A &= \frac{kr_3(r_2 - kb_2)}{r_2r_3 + k^2b_2(c_2-\gamma)}, \\ B &= \frac{k(r_3(b_1-\alpha) - kb_2(c_1-\beta))}{r_2r_3 + k^2b_2(c_2-\gamma)}\bar{u}, \\ C &= k\frac{(r_2r_3 + 2k^2b_2c_2 + kr_2(c_2-\gamma))}{r_2r_3 + kb_2(c_2-\gamma)}, \\ D &= k\frac{(2r_2(c_1-\beta) + k(c_2-\gamma)(c_1-\alpha))}{r_2r_3 + kb_2(c_2-\gamma)}\bar{u}, \end{aligned}$$

we get

$$\begin{aligned} u &= \frac{k}{r_1}\left(r_1 - a_1 w - a_2 z - \tau\right), \\ \Rightarrow \bar{u} &= \frac{k(r_1 - a_1A - a_2C - \tau)}{r_1 + ka_1B + ka_2D}, \end{aligned}$$

for biological reasons, $\bar{u}$ exists if its components are positive. That lead to $\bar{u} > 0$, $\bar{w} > 0$ and $\bar{z} > 0$. That to say

$$\begin{aligned} &A > 0, \quad B > 0, \quad C > 0, \\ &D > 0, \quad r_1 > a_1A + a_2C + \tau, \end{aligned}$$

it follows

$$\begin{aligned} & r_2 > b_2k, \quad c_2 > \gamma, \quad b_1 > \alpha, \\ & c_1 > \beta, \quad r_1 > a_1A + a_2C + \tau, \\ \textit{and} \quad & r_3r_2 + r_3b_1 + b_2\beta\bar{u} > r_3kb_2 + r_3\alpha + b_2c_1\bar{u}. \end{aligned}$$

■

3.4 Analysis of the stability of equilibrium points

We study the stability of equilibrium points of the system (76) by using the Jacobian matrix of the system and Routh-Hurwitz criterion.

3.4.1 Local stability of trivial and semi-trivial equilibrium points

The Jacobian matrix of the system (76) is given by

$$Jac(f) = \begin{pmatrix} a_{11} & -a_1\,u & -a_2\,u \\ (b_1-\alpha)\,w & a_{22} & -b_2\,w \\ (c_1-\beta)\,z & c_2\,z & a_{33} \end{pmatrix}, \tag{108}$$

where

$$\begin{aligned} a_{11} &= r_1 - \frac{2\,r_1}{k}\,u - a_1\,w - a_2\,z - \tau, \\ a_{22} &= r_2 - \frac{2\,r_2}{k}\,w + (b_1-\alpha)\,u - b_2\,z, \\ a_{33} &= r_3 - \frac{2\,r_3}{k}\,z + (c_1-\beta)\,u + (c_2-\gamma)\,w. \end{aligned}$$

Following propositions show behaviour of the equilibrium points E_0^*, E_1^*, E_2^* and E_3^*.

Proposition 3.1 *The equilibrium point E_0 of the system (76) is unstable.*

Proof: The Jacobian matrix of the system (76) at equilibrium point E_0 is

$$Jac(f)(E_0) = \begin{pmatrix} r_1-\tau & 0 & 0 \\ 0 & r_2 & 0 \\ 0 & 0 & r_3 \end{pmatrix}, \tag{109}$$

the eigenvalues are $r_1 - \tau$, r_2 and r_3. Since eigenvalues r_2 and r_3 are positive, the equilibrium point E_0 is unstable. ■

Proposition 3.2 *The equilibrium point E_1 of the system (76) is stable if*

$$\begin{aligned} \tau < r_1, \quad & r_1r_2 + kb_1r_1 + k\alpha\tau < k\alpha r_1 + kb_1\tau, \\ \textit{and} \quad & r_1r_3 + kc_1r_1 + k\beta\tau < k\beta r_1 + kc_1\tau. \end{aligned}$$

Proof: The Jacobian matrix of the system (76) at equilibrium point E_1 is

$$Jac(f)(E_1) = \begin{pmatrix} a_{11} & a_{12} & -a_2\,k(1-\dfrac{\tau}{r_1}) \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}, \tag{110}$$

with

$$\begin{aligned} a_{11} &= \tau - r_1, \quad a_{12} = -a_1\,k(1-\frac{\tau}{r_1}), \\ a_{22} &= r_2 + k\,(b_1-\alpha)\,(1-\frac{\tau}{r_1}), \\ a_{33} &= r_3 + k\,(c_1-\beta)\,(1-\frac{\tau}{r_1}), \end{aligned}$$

its eigenvalues are $\tau - r_1$, $r_2 + k\,(b_1-\alpha)\,(1-\frac{\tau}{r_1})$, $r_3 + k\,(c_1-\beta)\,(1-\frac{\tau}{r_1})$. The equilibrium point E_1 is stable if its eigenvalues are all negative. A calculation gives the result. ■

Proposition 3.3 *The equilibrium point E_2 of the system (76) is unstable if $c_2 > \gamma$ and stable if*

$$r_1 < a_1\,k + \tau \;\; \textit{and} \;\; \frac{r_3}{k} + c_2 < \gamma.$$

Proof: The Jacobian matrix of the system (76) at equilibrium point E_2 is

$$Jac(f)(E_2) = \begin{pmatrix} B_{11} & 0 & 0 \\ (b_1-\alpha)\,k & B_{22} & -b_2k \\ 0 & 0 & B_{33} \end{pmatrix}, \tag{111}$$

with

$$\begin{aligned} B_{11} &= r_1 - a_1\,k - \tau, \\ B_{22} &= -r_2, \\ B_{33} &= r_3 + (c_2 - \gamma)\;k. \end{aligned}$$

its eigenvalues are $r_1 - a_1\,k - \tau$, $-r_2$, $r_3 + (c_2-\gamma)\;k$ are not always of the same sign. If $c_2 > \gamma$ and $r_1 > a_1\,k + \tau$ then, the eigenvalues $r_1 - a_1\,k - \tau$ and $r_3 + (c_2-\gamma)\;k$ are positive. In this case the equilibrium point E_2 is unstable. Otherwise the equilibrium point E_2 is stable if its eigenvalues are all negative. A calculation gives the result. ■

Proposition 3.4 *The equilibrium point E_3 of the system (76) is unstable if $r_1 > a_2\,k$ and $r_2 > b_2\,k$ and stable if $r_1 < a_2\,k$ and $r_2 < b_2\,k$.*

Proof: The Jacobian matrix of the system (76) at equilibrium point E_3 is

$$Jac(f)(E_3) = \begin{pmatrix} r_1 - a_2\,k & 0 & 0 \\ 0 & r_2 - b_2\,k & 0 \\ (c_1-\beta)\;k & c_2\,k & -r_3 \end{pmatrix}, \tag{112}$$

and its eigenvalues are $r_1 - a_2\,k$, $r_2 - b_2\,k$ and $-r_3$. If $r_1 > a_2\,k$ and $r_2 > b_2\,k$ then, the eigenvalues $r_1 - a_2\,k$ and $r_2 - b_2\,k$ are positive. In this case the equilibrium point E_3 is unstable. Otherwise the equilibrium point E_3 is stable if its eigenvalues are all negative. That to say $r_1 < a_2\,k$ and $r_2 < b_2\,k$. ■

Proposition 3.5 *The equilibrium point E_4 of the system (76) is asymptotically stable if the conditions below are satisfied.*

$$\begin{aligned} & r_1\,(r_2 + b_1\,u^*) + \left(\frac{2\,r_1}{k}\,u^* + \tau\right)\left(\frac{2\,r_2}{k}\,w^* + \alpha\,u^*\right) \\ + \;& \frac{2r_2}{k} a_1 (w^*)^2 > r_1\left(\frac{2\,r_2}{k}\,w^* + \alpha\,u^*\right) + r_2 a_1 w^* \\ + \;& \left(\frac{2\,r_1}{k}\,u^* + \tau\right)(r_2 + b_1\,u^*), \end{aligned}$$

$$\begin{aligned} & \beta\,u^* + \gamma\,w^* > r_3 + c_1\,u^* + c_2\,w^* \\ \textit{and} \quad & \frac{2}{k}(b_1 + r_1)u^* + \alpha u^* + r_2 + a_1\,w^* + \tau > \\ & r_1 + b_1 u^* + \frac{2\alpha}{k} u^*. \end{aligned}$$

Proof: The Jacobian matrix of the system (76) at equilibrium point E_4 is

$$Jac(f)(E_4) = \begin{pmatrix} a_{11} & -a_1\,u^* & -a_2\,u^* \\ (b_1-\alpha)\;w^* & a_{22} & -b_2\,w^* \\ 0 & 0 & a_{33} \end{pmatrix}, \tag{113}$$

where

$$a_{11} = r_1 - \frac{2\,r_1}{k}\,u^* - a_1\,w^* - \tau, \tag{114}$$

$$a_{22} = r_2 - \frac{2\,r_2}{k}\,w^* + (b_1-\alpha)\;u^*, \tag{115}$$

$$a_{33} = r_3 + (c_1-\beta)\;u^* + (c_2-\gamma)\;w^*, \tag{116}$$

and the characteristic polynomial is

$$\begin{aligned} P_0(X) &= (a_{33} - X)\Big(X^2 - (a_{11} + a_{22})\,X \\ &+ \; a_{11}a_{22} + a_1\,(b_1-\alpha)\,u^* w^*\Big), \end{aligned} \tag{117}$$

according to Routh Hurwitz criterion, the equilibrium point E_4 is asymptotically stable if the roots of characteristic polynomial $P_0(X)$ have all negative real part. That is to say

$$X_0 = a_{33} < 0, \tag{118}$$

$$(a_{11} + a_{22}) < 0, \tag{119}$$

$$a_{11}a_{22} + a_1\,(b_1-\alpha)\,u^* w^* > 0, \tag{120}$$

then, we have

$$\begin{aligned} X_0 &= a_{33} < 0, \\ \Rightarrow \;& a_{33} = r_3 + (c_1-\beta)\;u^* + (c_2-\gamma)\;w^* < 0, \\ \Rightarrow \;& r_3 + c_1\,u^* - \beta\,u^* + c_2\,w^* - \gamma\,w^* < 0, \\ \Rightarrow \;& r_3 + c_1\,u^* + c_2\,w^* < \beta\,u^* + \gamma\,w^*, \end{aligned} \tag{121}$$

moreover, we have

$$\begin{aligned} a_{11} + a_{22} &= r_1 + r_2 + (b_1-\alpha)\;u^* - a_1\,w^* - \tau \\ &- \frac{2}{k}\,(r_1\,u^* + (b_1-\alpha)u^* + r_2 k), \\ &= r_1 - \frac{2}{k} r_1\,u^* + (b_1-\alpha)\;u^* \\ &- \frac{2}{k}(b_1-\alpha)u^* - r_2 - a_1\,w^* - \tau, \\ &= r_1(1 - \frac{2}{k}\,u^*) + (1 - \frac{2}{k})(b_1-\alpha)u^* \\ &- r_2 - a_1\,w^* - \tau, \end{aligned} \tag{122}$$

then,

$$\begin{aligned} & a_{11} + a_{22} < 0, \\ \Rightarrow \;& r_1 + b_1 u^* + \frac{2\alpha}{k} u^* < \frac{2}{k}(b_1 + r_1)u^* \\ + \;& \alpha u^* + r_2 + a_1\,w^* + \tau, \end{aligned}$$

and finally

$$\begin{aligned} & a_{11}a_{22} + a_1\,(b_1-\alpha)\,u^*w^* > 0, \\ \Rightarrow\ & r_1\,(r_2+b_1\,u^*) + \frac{2r_2}{k}a_1(w^*)^2 \\ + & \left(\frac{2\,r_1}{k}\,u^*+\tau\right)\left(\frac{2\,r_2}{k}\,w^*+\alpha\,u^*\right) \\ > & \; r_1\left(\frac{2\,r_2}{k}\,w^*+\alpha\,u^*\right) + r_2a_1w^* \\ + & \left(\frac{2\,r_1}{k}\,u^*+\tau\right)(r_2+b_1\,u^*)\,. \end{aligned} \tag{123}$$

Therefore, the equilibrium point E_4 is asymptotically stable if the conditions (121), (122) and (123) are satisfied. ■

Proposition 3.6 *The equilibrium point E_5 of the system (76) is asymptotically stable if the conditions below are satisfied.*

i)

$$\begin{aligned} & \left(\frac{2\,r_1}{k}\,u^*+\tau\right)\left(\frac{2\,r_3}{k}\,z^*+\beta\,u^*\right) \\ + & \frac{2r_3}{k}a_2(z^*)^2 + r_1\,(r_3+c_1\,u^*) > \\ & r_1\left(\frac{2\,r_3}{k}\,z^*+\beta\,u^*\right) + r_3a_2z^* \\ + & \left(\frac{2\,r_1}{k}\,u^*+\tau\right)(r_3+c_1\,u^*)\,, \end{aligned}$$

ii) $\alpha\,u^* + b_2\,z^* > r_2 + b_1\,u^*$,

iii)

$$\beta u^* + r_3 + a_2z^* + \tau + \frac{2(r_1+c_1)u^*}{k} > r_1 + c_1u^* + \frac{2\beta u^*}{k}.$$

Proof: The Jacobian matrix of the system (76) at equilibrium point E_5 is

$$Jac(f)(E_5) = \begin{pmatrix} a_{11} & -a_1\,u^* & -a_2\,u^* \\ 0 & a_{22} & 0 \\ (c_1-\beta)\,z^* & c_2\,z^* & a_{33} \end{pmatrix}, \tag{124}$$

where

$$\begin{aligned} a_{11} &= r_1 - \frac{2\,r_1}{k}\,u^* - a_2\,z^* - \tau, \\ a_{22} &= r_2 + (b_1-\alpha)\,u^* - b_2\,z^*, \\ a_{33} &= r_3 - \frac{2\,r_3}{k}\,z^* + (c_1-\beta)\,u^*, \end{aligned}$$

and its characteristic polynomial is

$$\begin{aligned} P_1(X) = (a_{22}-X)\Big(& X^2 - (a_{11}+a_{33})\,X \\ & + a_{11}a_{33} + a_2\,(c_1-\beta)\,u^*z^*\Big), \end{aligned} \tag{125}$$

according to Routh Hurwitz criterion, equilibrium point E_5 is asymptotically stable if the roots of characteristic polynomial $P_1(X)$ have all negative real part. That is to say

$$X_0 = a_{22} < 0, \tag{126}$$
$$a_{11}+a_{33} < 0, \tag{127}$$
$$a_{11}a_{33} + a_2\,(c_1-\beta)\,u^*z^* > 0, \tag{128}$$

then, we have

$$\begin{aligned} X_0 &= a_{22} < 0, \\ \Rightarrow\ & a_{22} = r_2 + (b_1-\alpha)\,u^* - b_2\,z^* < 0, \\ \Rightarrow\ & r_2 + b_1\,u^* < \alpha\,u^* + b_2\,z^*, \end{aligned} \tag{129}$$

moreover, we have

$$\begin{aligned} a_{11}+a_{33} &= r_1 - \frac{2\,r_1}{k}\,u^* - a_2\,z^* - \tau + r_3 \\ &- \frac{2\,r_3}{k}\,z^* + (c_1-\beta)\,u^*, \\ &= r_1(1-\frac{2}{k}\,u^*) + (1-\frac{2}{k})(c_1-\beta)u^* \\ &- r_3 - a_2\,z^* - \tau, \end{aligned}$$

then,

$$\begin{aligned} & a_{11}+a_{33} < 0, \\ \Rightarrow\ & r_1 + c_1u^* + \frac{2\beta u^*}{k} < \frac{2(r_1+c_1)u^*}{k} \\ + & \beta u^* + r_3 + a_2z^* + \tau, \end{aligned} \tag{130}$$

and finally

$$\begin{aligned} & a_{11}a_{33} + a_2\,(c_1-\beta)\,u^*z^* > 0, \\ \Rightarrow\ & r_1\,(r_3+c_1\,u^*) + \frac{2r_3}{k}a_2(z^*)^2 \\ + & \left(\frac{2\,r_1}{k}\,u^*+\tau\right)\left(\frac{2\,r_3}{k}\,z^*+\beta\,u^*\right) > \\ & r_1\left(\frac{2\,r_3}{k}\,z^*+\beta\,u^*\right) + r_3a_2z^* \\ + & \left(\frac{2\,r_1}{k}\,u^*+\tau\right)(r_3+c_1\,u^*)\,. \end{aligned} \tag{131}$$

Therefore, the equilibrium point E_5 is asymptotically stable if the conditions (129), (130) and (131) are satisfied. ■

Proposition 3.7 *The equilibrium point E_6 of the system (76) is asymptotically stable if the conditions below are satisfied.*

i)

$$\begin{aligned} r_2 &\quad (r_3 + c_2\, w^*) + \frac{2\, r_2}{k}\, w^* \left(\frac{2\, r_3}{k}\, z^* + \gamma\, w^*\right) \\ &+ \quad b_2\, z^* \left(\frac{2\, r_3}{k}\, z^* + \gamma\, w^*\right) > +r_3 b_2 z^* \\ &+ \quad r_2 \left(\frac{2\, r_3}{k}\, z^* + \gamma\, w^*\right) + \frac{2\, r_2}{k}\, w^* \,(r_3 + c_2\, w^*)\,, \end{aligned}$$

ii)

$$\alpha\, u^* + b_2\, z^* > r_2 + b_1\, u^*,$$

iii)

$$\begin{aligned} r_3 + b_2 z^* + \gamma w^* + \frac{2(c_2 + r_2)}{k} w^* \\ > r_2 + c_2 w^* + \frac{2\gamma}{k} w^*. \end{aligned}$$

Proof: The Jacobian matrix of the system (76) at equilibrium point E_6 is

$$Jac(f)(E_6) = \begin{pmatrix} a_{11} & 0 & 0 \\ (b_1 - \alpha)\; w^* & a_{22} & -b_2\, w^* \\ (c_1 - \beta)\; z^* & c_2\, z^* & a_{33} \end{pmatrix}, \tag{132}$$

where

$$\begin{aligned} a_{11} &= r_1 - a_1\, w^* - a_2\, z^*, && (133) \\ a_{22} &= r_2 - \frac{2\, r_2}{k}\, w^* - b_2\, z^*, && (134) \\ a_{33} &= r_3 - \frac{2\, r_3}{k}\, z^* + (c_2 - \gamma)\; w^*, && (135) \end{aligned}$$

and its characteristic polynomial is

$$\begin{aligned} P_2(X) = (a_{11} - X) \Big(X^2 - (a_{22} + a_{33})\, X \\ + a_{22}a_{33} + b_2\, c_2 w^* z^*\Big), \quad (136) \end{aligned}$$

according to Routh Hurwitz criterion, equilibrium point E_6 is asymptotically stable if the roots of characteristic polynomial $P_2(X)$ have all negative real part. That is to say

$$\begin{aligned} X_0 = a_{11} &< 0, && (137) \\ a_{22} + a_{33} &< 0, && (138) \\ a_{22}a_{33} + b_2\, c_2 w^* z^* &> 0, && (139) \end{aligned}$$

then, we have

$$\begin{aligned} X_0 &= a_{11} < 0 \\ \Rightarrow &\quad a_{11} = r_2 + (b_1 - \alpha)\; u^* - b_2\, z^* < 0 \\ \Rightarrow &\quad r_2 + b_1\, u^* < \alpha\, u^* + b_2\, z^*, \end{aligned} \tag{140}$$

moreover, we have

$$\begin{aligned} a_{22} + a_{33} &= r_2 + r_3 + (c_2 - \gamma)\; w^* - \frac{2\, r_2}{k}\, w^* \\ &- \quad \frac{2\, r_3}{k}\, z^* - b_2\, z^*, \\ &= r_2 (1 - \frac{2}{k}\, w^*) + (1 - \frac{2}{k})(c_2 - \gamma) w^* \\ &- \quad r_3 - b_2\, z^*, \end{aligned}$$

then,

$$\begin{aligned} & a_{22} + a_{33} < 0, \\ \Rightarrow &\quad r_2 + c_2 w^* + \frac{2\gamma}{k} w^* < r_3 + b_2 z^* \\ + &\quad \gamma w^* + \frac{2c_2}{k} w^* + \frac{2r_2}{k} w^*, \end{aligned} \tag{141}$$

and finally

$$\begin{aligned} & a_{22}a_{33} + b_2\, c_2 w^* z^* > 0, \\ \Rightarrow &\quad r_2\, (r_3 + c_2\, w^*) + \frac{2\, r_2}{k}\, w^* \left(\frac{2\, r_3}{k}\, z^* + \gamma\, w^*\right) \\ + &\quad b_2\, z^* \left(\frac{2\, r_3}{k}\, z^* + \gamma\, w^*\right) > r_3 b_2 z^* \\ + &\quad r_2 \left(\frac{2\, r_3}{k}\, z^* + \gamma\, w^*\right) + \frac{2\, r_2}{k}\, w^* \,(r_3 + c_2\, w^*)\,. \end{aligned} \tag{142}$$

Therefore, the equilibrium point E_6 is asymptotically stable if the conditions (140), (141) and (142) are satisfied. ■

3.4.2 Local stability of positive equilibrium point

let's put

$$\begin{aligned} m_1 &= r_1, \qquad m_2 = \frac{2\, r_1}{k}\, \bar{u} + a_1\, \bar{w} + a_2\, \bar{z} + \tau, \\ m_3 &= r_2 + (b_1 - \alpha)\,, \quad m_4 = \frac{2\, r_2}{k}\, \bar{w} + b_2\, \bar{z}, \\ m_5 &= r_3 + (c_1 - \beta)\; \bar{u} + (c_2 - \gamma)\; \bar{w}, \\ m_6 &= \frac{2\, r_3}{k}\, \bar{z}, \\ M_1 &= m_1 m_3 m_5 + m_2 m_4 m_5 \\ &+ \quad m_1 m_4 m_6 + m_2 m_3 m_6, \\ M_2 &= m_1 m_4 m_5 + m_2 m_3 m_5 \\ &+ \quad m_1 m_3 m_6 + m_2 m_4 m_6, \end{aligned}$$

$$\begin{aligned}\rho_1 &= 2M_2 + (m_4+m_6)(m_1-m_2)^2\\ &+ (m_2+m_6)(m_3-m_4)^2\\ &+ a_1(m_2+m_4)\,(b_1-\alpha)\;\bar u\,\bar w\\ &+ a_2(m_2+m_6)\,(c_1-\beta)\;\bar u\,\bar z\\ &+ (m_2+m_4)(m_5-m_6)^2\\ &+ b_2c_2(m_4+m_6)\,\bar w\,\bar z,\end{aligned}$$

$$\begin{aligned}\rho_2 &= 2M_1 + (m_3+m_5)(m_1-m_2)^2\\ &+ (m_1+m_5)^2(m_3-m_4)^2\\ &+ a_1(m_1+m_3)\,(b_1-\alpha)\;\bar u\,\bar w\\ &+ a_2(m_1+m_5)\,(c_1-\beta)\;\bar u\,\bar z\\ &+ (m_1+m_3)(m_5-m_6)^2\\ &+ b_2c_2(m_3+m_5)\,\bar w\,\bar z,\end{aligned}$$

$$\begin{aligned}\rho_3 &= M_2 + a_2\,m_4\,(c_1-\beta)\;\bar u\,\bar z + b_2c_2\,m_2\,\bar w\,\bar z\\ &+ a_1\,(b_1-\alpha)\;m_6\,\bar u\,\bar w + a_2c_2\,(b_1-\alpha)\;\bar u\,\bar w\,\bar z,\\ \rho_4 &= M_1 + a_2\,m_3\,(c_1-\beta)\;\bar u\,\bar z + b_2c_2\,m_1\,\bar w\,\bar z\\ &+ a_1\,(b_1-\alpha)\;m_5\,\bar u\,\bar w + a_1\,b_2\,(c_1-\beta)\;\bar u\,\bar w\,\bar z.\end{aligned}$$

Theorem 3.2 *Suppose the following assumptions are satisfied*

i) $\left(\frac{2\,r_1}{k}+\alpha+\beta\right)\bar u + \left(\frac{2\,r_2}{k}+a_1+\gamma\right)\bar w + \left(\frac{2\,r_3}{k}+a_2+b_2\right)\bar z + \tau > r_1+r_2+r_3+(b_1+c_1)\;\bar u + c_2\,\bar w.$

ii) $\rho_1 > \rho_2$.

iii) $\rho_3 > \rho_4$.

Then, the positive equilibrium point $\bar E = (\bar u, \bar w, \bar z)^T$ *of the system (76) is locally asymptotically stable.*

Proof: The Jacobian matrix of system (76) at the positive equilibrium point $\bar E = (\bar u, \bar w, \bar z)^T$ is

$$Jac(f)(\bar E) = \begin{pmatrix} a_{11} & -a_1\,\bar u & -a_2\,\bar u\\ (b_1-\alpha)\;\bar w & a_{22} & -b_2\,\bar w\\ (c_1-\beta)\;\bar z & c_2\,\bar z & a_{33}\end{pmatrix}, \tag{143}$$

where

$$\begin{aligned}a_{11} &= r_1 - \frac{2\,r_1}{k}\,\bar u - a_1\,\bar w - a_2\,\bar z - \tau,\\ a_{22} &= r_2 - \frac{2\,r_2}{k}\,\bar w + (b_1-\alpha)\;\bar u - b_2\,\bar z,\\ a_{33} &= r_3 - \frac{2\,r_3}{k}\,\bar z + (c_1-\beta)\;\bar u + (c_2-\gamma)\;\bar w,\end{aligned}$$

the characteristic polynomial of $Jac(f)(\bar E)$ is

$$P(X) = X^3 + \lambda_2 X^2 + \lambda_1 X + \lambda_0, \tag{144}$$

with

$$\begin{aligned}\lambda_2 &= -(a_{11}+a_{22}+a_{33}),\\ \lambda_1 &= a_{11}a_{22}+a_{11}a_{33}+a_{22}a_{33}\\ &- a_{12}a_{21}-a_{23}a_{32}-a_{13}a_{31},\\ \lambda_0 &= a_{13}a_{31}a_{22}+a_{23}a_{32}a_{11}+a_{12}a_{21}a_{33}\\ &- a_{11}a_{22}a_{33}-a_{12}a_{23}a_{31}-a_{13}a_{21}a_{32}.\end{aligned}$$

The equilibrium point $\bar E = (\bar u, \bar w, \bar z)^T$ is locally asymptotically stable, if the following Routh Hurwitz criterion is satisfied.

$$\lambda_2 > 0, \tag{145}$$

$$\lambda_2\,\lambda_1 - \lambda_0 > 0, \tag{146}$$

$$\lambda_0 > 0. \tag{147}$$

Inequality (145) leads to

$$\begin{aligned}\lambda_2 &> 0 \quad\Rightarrow\quad -(a_{11}+a_{22}+a_{33}) > 0,\\ \Rightarrow &\quad \left(\frac{2\,r_1}{k}+\alpha+\beta\right)\bar u + \left(\frac{2\,r_2}{k}+a_1+\gamma\right)\bar w\\ &+ \left(\frac{2\,r_3}{k}+a_2+b_2\right)\bar z + \tau > r_1+r_2+r_3\\ &+ (b_1+c_1)\;\bar u + c_2\,\bar w.\end{aligned}$$

Moreover, the inequality (146) leads to

$$\begin{aligned}&\lambda_2\,\lambda_1 - \lambda_0 > 0\\ \Rightarrow &- a_{11}a_{11}a_{22} - a_{11}a_{11}a_{33} - 2a_{11}a_{22}a_{33}\\ &+ a_{11}a_{12}a_{21} + a_{11}a_{13}a_{31} - a_{22}a_{11}a_{22}\\ &- a_{22}a_{22}a_{33} + a_{22}a_{12}a_{21} + a_{22}a_{23}a_{32}\\ &- a_{33}a_{11}a_{33} - a_{33}a_{22}a_{33} + a_{33}a_{23}a_{32}\\ &+ a_{33}a_{13}a_{31} + a_{12}a_{23}a_{31} + a_{13}a_{21}a_{32} > 0,\end{aligned}$$

using the above hypotheses, we have

$$\begin{array}{rl} & \lambda_2\,\lambda_1-\lambda_0>0 \\ \Rightarrow & \Big(2M_2+(m_4+m_6)(m_1-m_2)^2 \\ + & (m_2+m_6)(m_3-m_4)^2 \\ + & a_1(m_2+m_4)\,(b_1-\alpha)\;\bar u\,\bar w \\ + & a_2(m_2+m_6)\,(c_1-\beta)\;\bar u\,\bar z \\ + & (m_2+m_4)(m_5-m_6)^2 \\ + & b_2c_2(m_4+m_6)\,\bar w\,\bar z\Big) \\ > & \\ & \Big(2M_1+(m_3+m_5)(m_1-m_2)^2 \\ + & (m_1+m_5)^2(m_3-m_4)^2 \\ + & a_1(m_1+m_3)\,(b_1-\alpha)\;\bar u\,\bar w \\ + & a_2(m_1+m_5)\,(c_1-\beta)\;\bar u\,\bar z \\ + & (m_1+m_3)(m_5-m_6)^2 \\ + & b_2c_2(m_3+m_5)\,\bar w\,\bar z\Big), \end{array}$$

then, using the above hypotheses, we have

$$\lambda_2\,\lambda_1-\lambda_0\;>\;0\quad\Rightarrow\quad\rho_1>\rho_2. \tag{148}$$

Finally, inequality (147) leads to

$$\begin{array}{rcl} \lambda_0 & > & 0, \\ \lambda_0 & = & a_{13}a_{31}a_{22}+a_{23}a_{32}a_{11}+a_{12}a_{21}a_{33} \\ & - & a_{11}a_{22}a_{33}-a_{12}a_{23}a_{31}-a_{13}a_{21}a_{32}>0, \\ \lambda_0 & = & M_2+a_2\,m_4\,(c_1-\beta)\;\bar u\,\bar z+b_2c_2\,m_2\,\bar w\,\bar z \\ & + & a_1\,(b_1-\alpha)\;m_6\,\bar u\,\bar w+a_2c_2\,(b_1-\alpha)\;\bar u\,\bar w\,\bar z \\ & - & M_1-a_2\,m_3\,(c_1-\beta)\;\bar u\,\bar z-b_2c_2\,m_1\,\bar w\,\bar z \\ & - & a_1\,(b_1-\alpha)\;m_5\,\bar u\,\bar w-a_1\,b_2\,(c_1-\beta)\;\bar u\,\bar w\,\bar z, \end{array}$$

then, using the above hypotheses, we have

$$\lambda_0\;>\;0\quad\Rightarrow\quad\rho_3>\rho_4. \tag{149}$$

■

3.4.3 Global stability of positive equilibrium point

We study the global stability of the equilibrium point $\bar E=(\bar u,\bar w,\bar z)^T$ of system (76).

Theorem 3.3 *Suppose the following assumptions are satisfied,*

$$\begin{array}{rcl} r_1-\dfrac{r_1\bar u}{k} & = & a_1\bar w+a_2\bar z, \\ r_2-\dfrac{r_2\bar w}{k} & = & -\,(b_1-\alpha)\,\bar u+b_2\bar z, \\ r_3-\dfrac{r_3\bar z}{k} & = & -\,(c_1-\beta)\,\bar u-(c_2-\gamma)\,\bar w, \\ and\;\;\nu_1 & = & \dfrac{c_2-\gamma}{b_2},\quad \nu_2=\dfrac{c_1-\beta}{a_1},\quad \nu_3=1. \end{array}$$

Then, the positive equilibrium point $\bar E$ is globally asymptotically stable.

Proof: Let be the following Lyapunov function

$$L:\;\mathbb{R}^3\;\longrightarrow\;\mathbb{R}$$

defined by

$$L(u,w,z)=L_1(u,w,z)+L_2(u,w,z)+L_3(u,w,z), \tag{150}$$

with

$$\begin{array}{rcl} L_1(u,w,z) & = & \nu_1\left((u-\bar u)-\bar u\,ln(\dfrac{u}{\bar u})\right), \\ L_2(u,w,z) & = & \nu_2\left((w-\bar w)-\bar w\,ln(\dfrac{w}{\bar w})\right), \\ L_3(u,w,z) & = & \nu_3\left((z-\bar z)-\bar z\,ln(\dfrac{z}{\bar z})\right), \end{array}$$

where $\nu_i,\ i\in\{1,\cdots,3\}$ are positive constants to be determined in the following. L is defined and continuous on $Int(\mathbb{R}^3_+)$. Note that,

$$\begin{cases} L(\bar u,\bar w,\bar z)=0, \\ L(\bar u,\bar w,\bar z)\;>\;0\;for\;all\;(u,v,w)^T\;\in\;\Omega\setminus(\bar u,\bar w,\bar z)^T, \end{cases} \tag{151}$$

$(\bar u,\bar w,\bar z)^T$ is therefore a global minimum of L. Furthermore, all solutions of the system are bounded and converge to Ω. For t sufficiently large, we'll restrict our study to this set.

The orbital derivative of L, i.e, the derivative with respect to time t along the solutions of the system (38) is

$$\frac{dL}{dt}=\frac{dL_1}{dt}+\frac{dL_2}{dt}+\frac{dL_3}{dt}, \tag{152}$$

using the following expression,

$$\frac{dL_i}{dt}=\frac{dL_i}{du}\frac{du}{dt}+\frac{dL_i}{dv}\frac{dv}{dt}+\frac{dL_i}{dw}\frac{dw}{dt}, \tag{153}$$

we have

$$\begin{array}{rcl} \dfrac{dL_1}{dt} & = & \nu_1\,(u-\bar u)\Big(r_1-\dfrac{r_1u}{k}-a_1w-a_2z\Big), \\ \dfrac{dL_2}{dt} & = & \nu_2\,(w-\bar w)\Big(r_2-\dfrac{r_2w}{k}+(b_1-\alpha)\,u-b_2z\Big), \\ \dfrac{dL_3}{dt} & = & \nu_3\,(z-\bar z)\times \\ & & \Big(r_3-\dfrac{r_3z}{k}+(c_1-\beta)\,u+(c_2-\gamma)\,w\Big), \end{array}$$

let's put

$$\begin{aligned} r_1 - \frac{r_1 \bar{u}}{k} &= a_1 \bar{w} + a_2 \bar{z}, \\ r_2 - \frac{r_2 \bar{w}}{k} &= -(b_1 - \alpha)\,\bar{u} + b_2 \bar{z}, \\ r_3 - \frac{r_3 \bar{z}}{k} &= -(c_1 - \beta)\,\bar{u} - (c_2 - \gamma)\,\bar{w}, \\ \textit{and}\ \ \nu_1 &= \frac{c_2 - \gamma}{b_2},\ \ \nu_2 = \frac{c_1 - \beta}{a_1},\ \ \nu_3 = 1. \end{aligned}$$

Then,

$$\begin{aligned} \frac{dL_1}{dt} &= \nu_1 (u - \bar{u})\Big(-\frac{r_1}{k}(u - \bar{u}) - a_1 (w - \bar{w}) \Big) \\ &- \nu_1 a_2 (u - \bar{u})(z - \bar{z}), \\ \frac{dL_2}{dt} &= \nu_2 (w - \bar{w})\Big(-\frac{r_2}{k}(w - \bar{w}) - b_2 (z - \bar{z}) \Big) \\ &- \nu_2 (w - \bar{w})(b_1 - \alpha)(u - \bar{u}), \\ \frac{dL_3}{dt} &= \nu_3 (z - \bar{z})\Big((c_1 - \beta)(u - \bar{u}) \Big) \\ +\nu_3 (z &- \bar{z})\Big((c_2 - \gamma)(w - \bar{w}) - \frac{r_3}{k}(z - \bar{z}) \Big), \end{aligned}$$

$\Rightarrow$

$$\begin{aligned} \frac{dL_1}{dt} &= -\frac{\nu_1 r_1}{k}(u - \bar{u})^2 - a_1 \nu_1 (u - \bar{u})(w - \bar{w}) \\ &- a_2 \nu_1 (u - \bar{u})(z - \bar{z}), \\ \frac{dL_2}{dt} &= -\frac{\nu_2 r_2}{k}(w - \bar{w})^2 - b_2 \nu_2 (w - \bar{w})(z - \bar{z}) \\ &- \nu_2 (b_1 - \alpha)(u - \bar{u})(w - \bar{w}), \\ \frac{dL_3}{dt} &= -\frac{\nu_3 r_3}{k}(z - \bar{z})^2 + \nu_3 (c_1 - \beta)(u - \bar{u})(z - \bar{z}) \\ &+ \nu_3 (c_2 - \gamma)(w - \bar{w})(z - \bar{z}), \end{aligned}$$

using expression (152), we have

$$\begin{aligned} \frac{dL}{dt} &= -\frac{\nu_1 r_1}{k}(u - \bar{u})^2 - \frac{\nu_2 r_2}{k}(w - \bar{w})^2 \\ &+ (\alpha \nu_2 - a_1 \nu_1 - b_1 \nu_2)(u - \bar{u})(w - \bar{w}) \\ &+ (c_1 \nu_3 - a_2 \nu_1 - \beta \nu_3)(u - \bar{u})(z - \bar{z}) \\ &+ (c_2 \nu_3 - b_1 \nu_2 - \gamma \nu_2)(w - \bar{w})(z - \bar{z}) \\ &- \frac{\nu_3 r_3}{k}(z - \bar{z})^2, \\ \frac{dL}{dt} &\leq -\frac{\nu_1 r_1}{k}(u - \bar{u})^2 - \frac{\nu_2 r_2}{k}(w - \bar{w})^2 \\ &- \frac{\nu_3 r_3}{k}(z - \bar{z})^2 < 0. \end{aligned}$$

Therefore, $\frac{dL}{dt} < 0$ and $\frac{dL}{dt} = 0$ iff $(u, v, w)^T = (\bar{u}, \bar{w}, \bar{z})^T$. So, L is a strict Lyapunov function and by LaSalle's invariance theorem, it follows that $(\bar{u}, \bar{w}, \bar{z})^T$ is globally asymptotically stable on Ω. ■

3.4.4 System permanence

System permanence addresses the problem of long-term population survival. The interest here is to find the conditions under which the interacting species in the system (76) will reach some form of coexistence over time. Let's put

$$\begin{aligned} \theta_1 &= a_1 \delta_2^* + a_2\, \delta_3^*, \\ \theta_2 &= \alpha_1 \delta_1^* + b_2\, \delta_3^*, \\ \theta_3 &= (c_1 - \beta)\, \delta_1^* + (c_2 - \gamma)\, \delta_2^*, \end{aligned}$$

with

$$\frac{r_3 + \theta_3}{r_3}\, k := \delta_3^*, \quad \frac{r_2 - \theta_2}{r_2}\, k := \delta_2^*, \quad \frac{r_1 - \theta_1}{r_1}\, k := \delta_1^*.$$

Definition 3.4 *[12]*
System (76) is said to be permanent if there exist positive constants m_i and M_i, $i \in \{1, \ldots, 3\}$ such that, for each component of the positive solution $X(t)$ of the system (76), we have

$$\begin{aligned} m_1 \leq \liminf_{t \longrightarrow +\infty} u(t) \leq \limsup_{t \longrightarrow +\infty} u(t) \leq M_1, \\ m_2 \leq \liminf_{t \longrightarrow +\infty} w(t) \leq \limsup_{t \longrightarrow +\infty} w(t) \leq M_2, \\ m_3 \leq \liminf_{t \longrightarrow +\infty} z(t) \leq \limsup_{t \longrightarrow +\infty} z(t) \leq M_3. \end{aligned}$$

Lemma 3.5 *[12]*
If there exist positive constants $M > 0$ and $n > 0$, such that for any positive solution $X(t)$ of the system (76), we have

$$n \leq \liminf_{t \longrightarrow +\infty} \|X(t)\| \leq \limsup_{t \longrightarrow +\infty} \|X(t)\| \leq M,$$

then, the system (76) is called uniformly permanent.

Theorem 3.4 *If hypotheses below are satisfied, then the system (76) is permanent.*

$$r_1 > \theta_1 \delta_2^* + a_2\, \delta_3^*, \tag{154}$$
$$r_2 + b_1 \delta_1^* > \theta_2, \tag{155}$$
$$r_3 + c_1 \delta_1^* + c_2\, \delta_2^* > \beta \delta_1^* + \gamma\, \delta_2^*. \tag{156}$$

Proof: According to Theorem 3.1 for any solution $X(t)$, we have the following inequalities

$$\limsup_{t \longrightarrow +\infty} u(t) \leq \delta_1 < +\infty, \tag{157}$$
$$\limsup_{t \longrightarrow +\infty} v(t) \leq \delta_2 < +\infty, \tag{158}$$
$$\limsup_{t \longrightarrow +\infty} w(t) \leq \delta_3 < +\infty. \tag{159}$$

There is a constant $M < +\infty$, $M = \max_i \delta_i$ such that $\forall\ i\ \in \{1, \dots, 3\}$, we have

$$\limsup_{t \longrightarrow +\infty} \|X(t)\| \leq M. \quad (160)$$

Let's prove that there exists $m > 0$ such that

$$\liminf_{t \longrightarrow +\infty} \|X(t)\| \geq m, \quad (161)$$

considering the equation $(76)_1$, we have

$$\begin{aligned} \frac{du}{dt} &= \Big(r_1 - \frac{r_1}{k}\,u - a_1 w - a_2\,z - \tau\Big)u, && (162)\\ \frac{du}{dt} &\geq \Big(r_1 - \frac{r_1}{k}\,u - a_1\delta_2 - a_2\,\delta_3\Big)u, && (163)\\ \frac{du}{dt} &\geq \Big(r_1 - \theta_1 - \frac{r_1}{k}\,u\Big)u. && (164) \end{aligned}$$

Using lemma 2.1, we have

$$\liminf_{t \longrightarrow +\infty} u(t) \geq \frac{r_1 - \theta_1}{r_1}\,k := \delta_1^*, \quad (165)$$

$\forall \epsilon_1 > 0$, there exists $T_1 > 0$ such that

$$u(t) > \delta_1^* - \epsilon_1\ \forall\ t \geq T_1. \quad (166)$$

By the same process, considering the equation $(76)_2$, we get

$$\liminf_{t \longrightarrow +\infty} w(t) \geq \frac{r_2 - \theta_2}{r_2}\,k := \delta_2^*. \quad (167)$$

We have for all $\epsilon_2 > 0$, there exists $T_2 > 0$ such that

$$w(t) > \delta_2^* - \epsilon_2\ \ \forall\ t \geq T_2,$$

$$with \quad \frac{r_2 + b_1\delta_1^* - \theta_2}{r_2}\,k := \delta_2^*.$$

Considering equation $(76)_3$, we have

$$\begin{aligned} \frac{dz}{dt} &= \Big(r_3 - \frac{r_3}{k}\,z + (c_1 - \beta)\ u + (c_2 - \gamma)\ w\Big)z,\\ \frac{dz}{dt} &\geq \Big(r_3 - \frac{r_3}{k}\,z + (c_1 - \beta)\,(\delta_1^* - \epsilon_1)\\ &+ \ (c_2 - \gamma)\ (\delta_2^* - \epsilon_2)\Big)z,\\ \frac{dz}{dt} &\geq \Big(r_3 + \theta - \frac{r_3}{k}\,z\Big)z. && (168) \end{aligned}$$

Then, by applying lemma 2.1 we obtain that

$$\liminf_{t \longrightarrow +\infty} z(t) \geq \frac{r_3 + \theta_3}{r_3}\,k\ := \delta_3^*, \quad (169)$$

$\forall \epsilon_3 > 0$, there exists $T_3 > 0$ such that

$$z(t) > \delta_3^* - \epsilon_3 \qquad \forall t \geq T_3. \quad (170)$$

Let $0 < m = \min_i \delta_i^*$, $\forall i \in \{1, \dots, 3\}$ then, according to the hypotheses of theorem 3.4 any solution $X(t)$ of system (76) verifies the inequation below

$$m \leq \liminf_{t \longrightarrow +\infty} \|X(t)\|. \quad (171)$$

■

3.4.5 Extinction of species

We're looking for the conditions under which the system (76) turns off over time. In fact, the extinction of the system determines the death or disappearance of all the species interacting in the system (76).

Definition 3.5 *[21] System (76) turns off, if*

$$\liminf_{t \longrightarrow +\infty} \|X(t)\| = 0. \quad (172)$$

Theorem 3.5 *Suppose conditions below are satisfied.*

i) $r_1 < a_1\delta_2^* + a_2\,\delta_3^*$.

ii) $r_2 + b_1\delta_1 < b_2\,\delta_3^* + \alpha\,\delta_1^*$.

iii) $r_3 + c_1\,\delta_1^* + c_2\,\delta_2^* < \gamma\,\delta_2^* + \beta\,\delta_1^*$.

Then, system (76) goes extinct, i.e, there is extinction of all interacting populations.

Proof: From equation $(76)_1$, we have

$$\begin{aligned} \frac{du}{dt} &= u\Big(r_1 - \frac{r_1 u}{k} - a_1 w - a_2\,z - \tau\Big), && (173)\\ \frac{du}{dt} &\leq u\Big(r_1 - a_1\delta_2^* - a_2\,\delta_3^*\Big), && (174)\\ u(t) &\leq u(0)exp\Big\{(r_1 - a_1\delta_2^* - a_2\,\delta_3^*)t\Big\}. && (175) \end{aligned}$$

Thus according to i), $u(t) \to 0$ when $t \to +\infty$. There is extinction of the population u. Likewise, from equation $(76)_2$, we obtain

$$w(t) \ \leq \ w(0)exp\{(r_2 + b_1\delta_1 - b_2\,\delta_3^* - \alpha\,\delta_1^*)t\}.$$

and if condition ii) is satisfied $w(t) \to 0$ when $t \to +\infty$. There is extinction of species w.

Considering equation $(76)_3$, we have

$$\begin{aligned} \frac{dz}{dt} &= \Big(r_3 - \frac{r_3}{k}\,z + (c_1 - \beta)\ u + (c_2 - \gamma)\ w\Big)z,\\ \frac{dz}{dt} &\leq z\Big(r_3 + c_1\,\delta_1^* + c_2\,\delta_2^* - \gamma\,\delta_2^* - \beta\,\delta_1^*\Big),\\ z(t) &\leq z(0)exp\Big\{(r_3 + c_1\,\delta_1^* + c_2\,\delta_2^* - \gamma\,\delta_2^* - \beta\,\delta_1^*)t\Big\}. \end{aligned}$$

Thus according to iii), $z(t) \to 0$ when $t \to +\infty$. There is extinction of predators z. ■

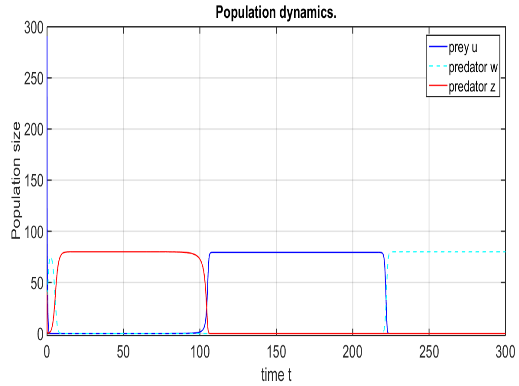
3.5 Numerical simulations

In this section, we verify the theoretical results. The parameters that are defined the in Tables 1 and 3, lead to Figures 15 and 16. Moreover, the Tables 2 and 4 show the stability of the equilibrium points. In addition, the Table 5 contains the parameters for the different phase portraits shown in Figure 17. These figures are discussed to help understand them.

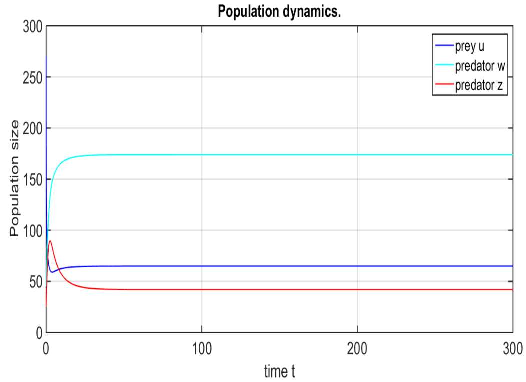
3.5.1 Evolution of interacting populations according to the model (76)

Model (76) presents a rich and varied dynamic. For initial conditions $(u_0; w_0; z_0)$ and different parameter values defined in the table 1, we obtain figures on Figure 15. We first obtain an evolution similar to that obtained in the model (12) for $(u_0; w_0; z_0) = (291; 30; 41)$. It's an evolution in the form of a continuous cycle in which each species seems to be the sole survivor while the other two are close to extinction. This evolution can be seen in figure 15a. Figure 15b shows a dynamic in which the three densities are stabilizing around $u \simeq 70$ for the preys, $w \simeq 175$ for intermediate predators and $z \simeq 49$. In figure 15c, densities have a decreasing aperiodic trend between $(u_{min}; w_{min}; z_{min}) \simeq (10.13; 4; 16.65)$ and $(u_{max}; w_{max}; z_{max}) \simeq (82.64; 74; 21.7)$ before stabilizing from $t \simeq 150$ at $u_{stable} \simeq 46.8$ for preys, $w_{stable} \simeq 13.2$ for intermediate predators and $z_{stable} \simeq 46.8$ for the super predators.

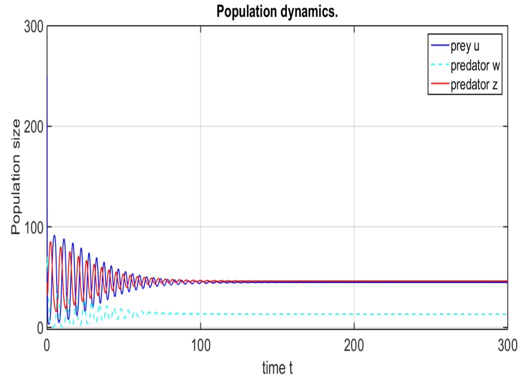
Figure 16 shows the dynamics in which species densities become extinct following parameters selected from Table 4. In Figure 16a, super-predators adapt to prey scarcity, while in Figure 16b, intermediate predators survive without direct predators. It should be noted that high predation rates are detrimental to predators. In fact, High consumption of infected prey inflicts losses on predator populations.



(a) Population's evolution with $(u_0 = 291, w_0 = 30, z_0 = 41)$



(b) Population's evolution with $(u_0 = 251, w_0 = 60, z_0 = 45)$

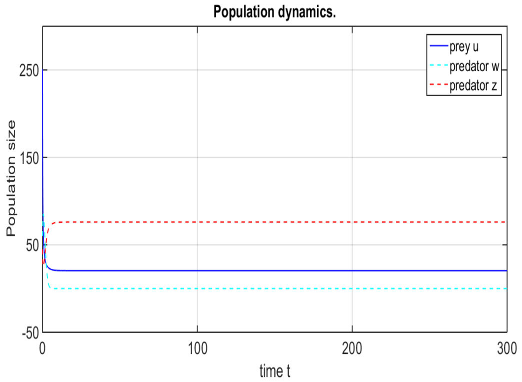


(c) Population's evolution with $(u_0 = 251, w_0 = 60, z_0 = 45)$

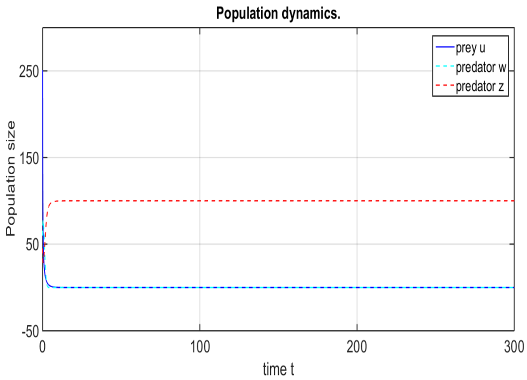
Figure 15: Some types of population evolution according to model (76).

3.5.2 Phase portraits for interacting populations according to the model (76)

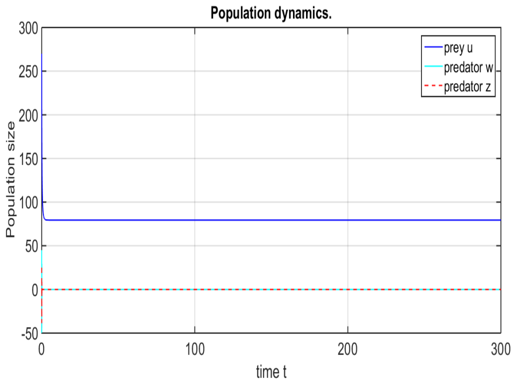
In Figure 17, we present several phase portraits to visualize the evolution of the different trajectories of the various densities in relation to each other. First, in Figure 17a for $(u_0; w_0; z_0) = (51;\ 20;\ 15)$, we obtain a coevolution of the three species over the time, the different species maintain stable population densities. Nevertheless, it should be noted that the density of super-predators is much higher than the other two densities, this is because predation rates are high. Furthermore, Figure 17b shows a similar dynamic in which the three densities are stabilizing their population densities. But, since predation rates



(a) Population's evolution with $(u_0 = 291, w_0 = 30, z_0 = 41)$



(b) Population's evolution with $(u_0 = 251, w_0 = 60, z_0 = 45)$



(c) Population's evolution with $(u_0 = 270, w_0 = 45, z_0 = 25)$

Figure 16: Some types of population extinction according to model (76).

Table 1: Input data for numerical simulations of Figure 15.

$Figure15(a)$	$Figure15(b)$	$Figure15(c)$
$r_1 = 2.71$	$r_1 = 2.71$	$r_1 = 2.71$
$r_2 = 1.94$	$r_2 = 1.94$	$r_2 = 1.94$
$r_3 = 0.75$	$r_3 = 1.75$	$r_3 = 1.75$
$a_1 = 0.101$	$a_1 = 0.00101$	$a_1 = 0.101$
$a_2 = 0.031$	$a_2 = 0.0031$	$a_2 = 0.0031$
$b_1 = 0.017$	$b_1 = 0.00101$	$b_1 = 0.017$
$b_2 = 0.052$	$b_2 = 0.0521$	$b_2 = 0.052$
$c_1 = 0.0075$	$c_1 = 0.00101$	$c_1 = 0.00101$
$c_2 = 0.017$	$c_2 = 0.0017$	$c_2 = 0.0017$
$\alpha = 0.0009$	$\alpha = 0.00109$	$\alpha = 0.00109$
$\beta = 0.073$	$\beta = 0.0215$	$\beta = 0.02197$
$\gamma = 0.012$	$\gamma = 0.00021$	$\gamma = 0.0021$
$k = 80.00$	$k = 100.00$	$k = 100.00$
$\tau = 0.0201$	$\tau = 0.921$	$\tau = 0.021$

Table 2: Equilibrium stability table for the data used in figure 15.

$Figure15(a)$		$Figure15(b)$		$Figure15(c)$	
E_0	*unstable*	E_0	*unstable*	E_0	*unstable*
E_1	*unstable*	E_1	*unstable*	E_1	*unstable*
E_2	*unstable*	E_2	*unstable*	E_2	*unstable*
E_3	*unstable*	E_3	*unstable*	E_3	*unstable*
E_4	×	E_4	*unstable*	E_4	×
E_5	*unstable*	E_5	*stable*	E_5	*stable*
E_6	×	E_6	×	E_6	×
$\bar{E}$	×	$\bar{E}$	*stable*	$\bar{E}$	×

Table 3: Input data for numerical simulations of Figure 16.

$Figure16(a)$	$Figure16(b)$	$Figure16(c)$
$r_1 = 2.59$	$r_1 = 2.59$	$r_1 = 2.59$
$r_2 = 0.5$	$r_2 = 0.5$	$r_2 = 0.5$
$r_3 = 0.420$	$r_3 = 0.42$	$r_3 = 0.42$
$a_1 = 0.001$	$a_1 = 0.001$	$a_1 = 0.001$
$a_2 = 0.004$	$a_2 = 0.004$	$a_2 = 0.004$
$b_1 = 0.471$	$b_1 = 0.0471$	$b_1 = 0.0471$
$b_2 = 0.001$	$b_2 = 0.001$	$b_2 = 0.001$
$c_1 = 0.041975$	$c_1 = 0.041975$	$c_1 = 0.041975$
$c_2 = 0.032971$	$c_2 = 0.03297$	$c_2 = 0.03297$
$\alpha = 0.29710$	$\alpha = 0.20319$	$\alpha = 0.22971$
$\beta = 0.020319$	$\beta = 0.02038$	$\beta = 0.20319$
$\gamma = 0.6700$	$\gamma = 0.00021$	$\gamma = 0.2038$
$k = 110.00$	$k = 110.00$	$k = 110.00$
$\tau = 0.71$	$\tau = 2.921$	$\tau = 0.7204$

have decreased slightly the density of intermediate predators seems to match that of super predators. However, for Figure 17c and Figure 17d, we slightly decrease predation rates and increase infection rate τ. As a result, the density of super predators has been severely disrupted and they are now on the brink of extinction. The increasing of infection rate seem to have serious consequences for populations densities.

Table 4: Equilibrium stability table for the data used in figure 16.

$Figure16(a)$		$Figure16(b)$		$Figure16(c)$	
E_0	$unstable$	E_0	$unstable$	E_0	$unstable$
E_1	$unstable$	E_1	$unstable$	E_1	$unstable$
E_2	$unstable$	E_2	$unstable$	E_2	$unstable$
E_3	$unstable$	E_3	$unstable$	E_3	$unstable$
E_4	$stable$	E_4	$unstable$	E_4	$unstable$
E_5	$unstable$	E_5	$unstable$	E_5	$unstable$
E_6	$unstable$	E_6	$unstable$	E_6	$unstable$
$\bar{E}$	$unstable$	$\bar{E}$	$unstable$	$\bar{E}$	$unstable$

Table 5: Input data for numerical simulations of Figure 17.

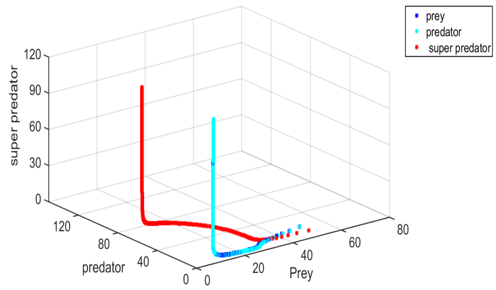
$Figure17(a)$	$Figure17(b)$	$Figure17(c)$	$Figure17(d)$
$r_1 = 2.71$	$r_1 = 2.71$	$r_1 = 2.95$	$r_1 = 2.05$
$r_2 = 0.94$	$r_2 = 1.94$	$r_2 = 1.15$	$r_2 = 0.50$
$r_3 = 1.5$	$r_3 = 1.75$	$r_3 = 1.021$	$r_3 = 0.020$
$a_1 = 0.101$	$a_1 = 0.101$	$a_1 = 0.071$	$a_1 = 0.001$
$a_2 = 0.031$	$a_2 = 0.0031$	$a_2 = 0.05$	$a_2 = 0.0041$
$b_1 = 0.00171$	$b_1 = 0.017$	$b_1 = 0.41$	$b_1 = 0.471$
$b_2 = 0.051$	$b_2 = 0.052$	$b_2 = 0.51$	$b_2 = 0.721$
$c_1 = 0.0075$	$c_1 = 0.00101$	$c_1 = 0.730$	$c_1 = 0.0475$
$c_2 = 0.017$	$c_2 = 0.0017$	$c_2 = 0.041$	$c_2 = 0.001$
$\alpha = 0.009$	$\alpha = 0.0109$	$\alpha = 0.970$	$\alpha = 0.97$
$\beta = 0.073$	$\beta = 0.02197$	$\beta = 0.035$	$\beta = 0.0365$
$\gamma = 0.012$	$\gamma = 0.021$	$\gamma = 0.480$	$\gamma = 0.670$
$k = 70$	$k = 70$	$k = 70$	$k = 11.074$
$\tau = 0.0201$	$\tau = 0.0212$	$\tau = 0.71$	$\tau = 0.71$

3.5.3 Some fields of application of the model (76)

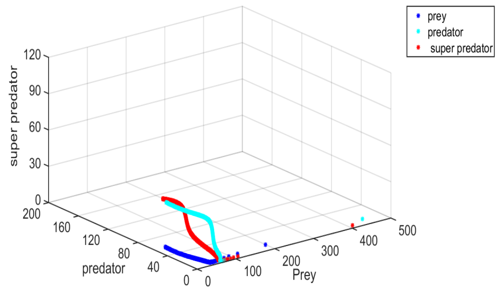
Mathematical models can find several fields of application. The model (12) for example and several of its variants have been studied with a view to their application to various fields.

As examples of fields of application, these models have been used in biology to study the dynamics of certain populations behaving in particular ways in given environments. In [22], authors show that a species of side-blotched lizards presents a heteroclinic cycle behavior. Populations alternately play the role of dominant species before being supplanted by a more competitive population. Other applications have been proposed in computational neuroscience, [23] and a stochastic extension of May-Leonard model has been appied to a neuromotor central pattern generator system, [24].

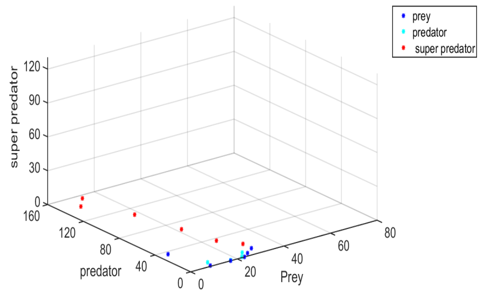
Our model presents a heteroclinic cycle behavior, very similar to the May-Leonard model, for certain parameter values. Therefore, the system could have applications in various fields. For example, we could study an extension for this model in the case where species evolve in a stochastic



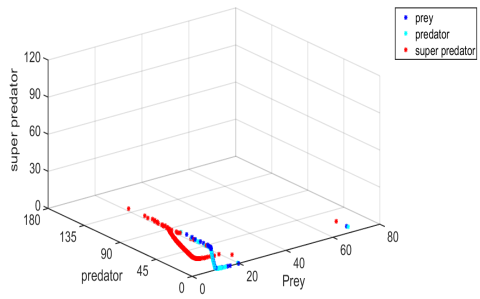
(a) The phase portrait with $(u_0 = 51, w_0 = 20, z_0 = 15)$



(b) The phase portrait with $(u_0 = 200, w_0 = 12, z_0 = 51)$



(c) The phase portrait with $(u_0 = 30, w_0 = 10, z_0 = 10)$



(d) The phase portrait with $(u_0 = 70, w_0 = 12, z_0 = 25)$

Figure 17: Some phase portrait according to the model (76).

environment. Furthermore, it could be useful in biology to study the evolution of a population attacked by a disease. The extensions could also be applied to computer science and control engineering.

4 Conclusion

In this article, we have explored several population dynamics models. The models studied highlight various types of interactions among three species within a natural environment. Our contribution in this paper can be summarized in two main stages.

First, we have analyzed various biological models of three interacting species. For each model, we determined the various equilibrium points, studied their stability and produced numerical illustrations using Matlab software to verify the results obtained. These include the May-Leonard three-competitor model and some three-species prey-predator models:

- A two-prey, one-predator model in which numerical tests have shown that species densities become stable after a certain time.
- A super-predator, predator and prey model in which numerical simulations have shown that prey, predator and super-predator population densities oscillate aperiodically before reaching to a steady state. Additionaly, for certain predation rates, populations can become extinct.
- A prey-predator model with prey harvesting, in which numerical simulations showed that the system can achieve stability depending of the prey harvesting rates.

In the second stage, we proposed and investigated a new model involving three species interacting in a natural environment. For this model, we took into account the notion of self-defense, with prey having within them certain infected individuals. An in-depth analysis of the model has allowed us to determine the conditions for both extinction and long-term survival of the species. This analysis has involved identifying various equilibrium points and studying their stability. Finally, we carried out numerical tests in a Maltab environment to illustrate population dynamics according to the different models presented. The simulation results showed a relationship between predation rate and a reduction of the predator population: as the predaton rate increased, the predator rate due to infection within the prey population also increased.

For future research, we plan to apply our model to various fields, particullary in medecine for controlling bacterial contamination in human, animal and plant population in agriculture also an animal and fishery resource management project for followed population growth will be considered. It should be added that we have outlined several areas in section 3.5.3 in which our work could be put to practical use in the near future.

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